

Work in small groups (3-4 people). Do NOT divide up the problems and work on them separately. **Work together** on all problems by sharing insights and difficulties, but **each student must hand in a set of solutions.** Ask your TA if you need help.

1. **Determine the domain and range** of the composition  $f \circ g$  for the case

$$f(x) = x^2, g(x) = -1 - \sqrt{x}.$$

**SOLUTION:**

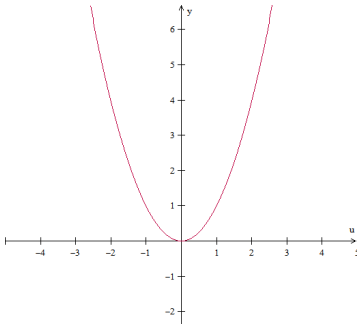
Please see the supplementary note posted in WebCT first; it should be useful.

Note that, with  $f(x) = x^2, g(x) = -1 - \sqrt{x}$ , we have:  $\text{dom}(f) = \mathbb{R}$  (the set of real numbers)  $\text{dom}(g) = [0, +\infty)$ .

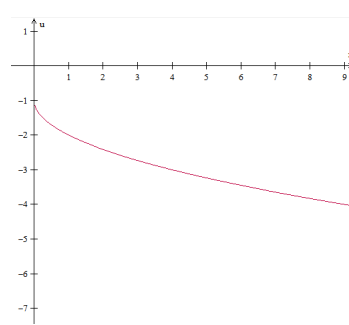
Also, from the graph of  $f$  and  $g$  we get:  $\text{ran}(f) = [0, +\infty)$  and  $\text{ran}(g) = (-\infty, -1)$

The graphs confirm the previous results:

$$f(u) = u^2$$



$$g(x) = -1 - \sqrt{x}$$



**Domain of the composition:**

$$\text{dom}(f \circ g) = \{x \in \text{dom}(g) \mid g(x) \in \text{dom}(f)\} =$$

$$\{x \in [0, +\infty) \mid g(x) \in \mathbb{R}\} = [0, +\infty)$$

So,  $\text{dom}(f \circ g) = [0, +\infty)$

**Range of the composition:**

$$\text{ran}(f \circ g) = \{y = f(g(x)) \mid g(x) \in \text{dom}(f) \cap \text{ran}(g)\} =$$

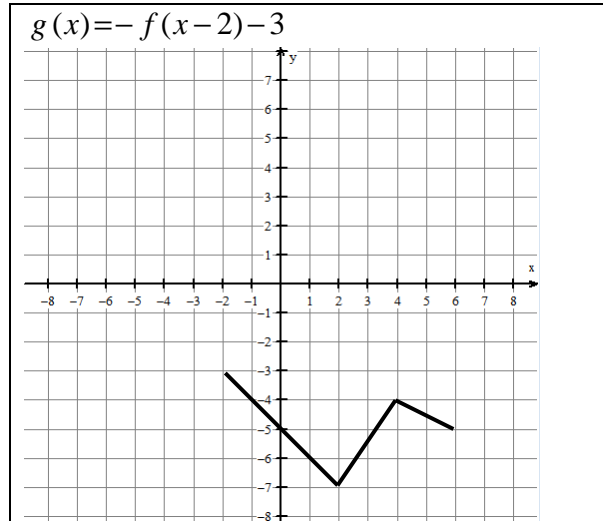
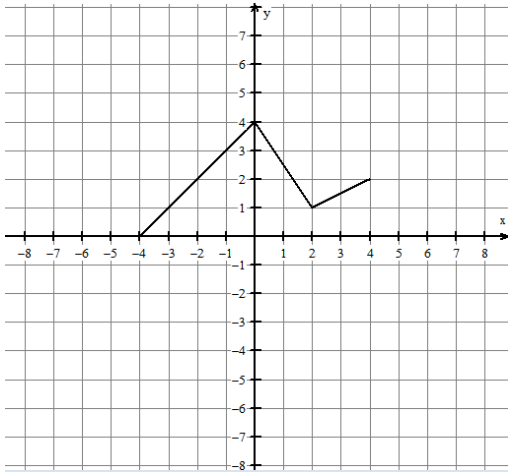
$$\{f(g(x)) \mid g(x) \in \mathbb{R} \cap (-\infty, -1)\} = \{f(u) \mid u \in (-\infty, -1)\} = \{u^2 \mid u \in (-\infty, -1)\} =$$

$$\dots = [1, +\infty)$$

So,  $\text{ran}(f \circ g) = [1, +\infty)$

This is obtained from the graphs, per the supplementary note posted in WebCT, under “Extra material”. As mentioned in the note, we will develop a systematic, ‘algebraic’ method to determine the range of a function in a given interval using derivatives.

2. The graph of a function  $f$  is given. Plot the graphs of the following function:



3. Solve the following for  $x$ , given that  $0 \leq x \leq 2\pi$  OR  $x \in [0, 2\pi]$ :

$$2 \cos x + \sin 2x = 0$$

**SOLUTION:**

$$2 \cos x + \sin 2x = 0 \quad 2 \cos x + 2 \sin x \cos x = 0 \quad 2 \cos x(1 + \sin x) = 0$$

$$\cos x = 0 \rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \quad 1 + \sin x = 0 \rightarrow \sin x = -1 \rightarrow x = \frac{3\pi}{2}$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$

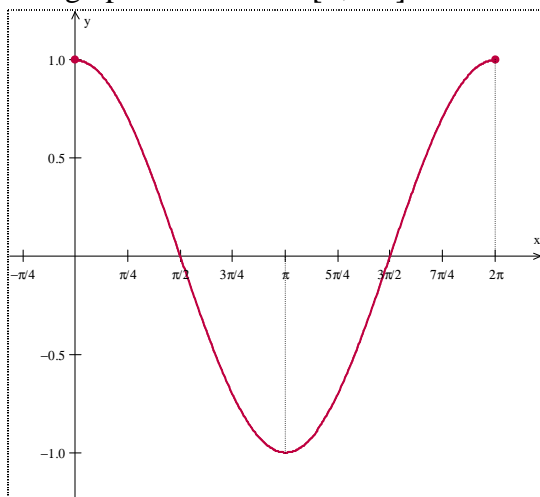
4. Without using a calculator, solve the following equation for  $t$ , where  $0 \leq t \leq 2\pi$  OR  $t \in [0, 2\pi]$ . Do not use a calculator, but recall the graphs of sin and cos and recall their periodic nature.

|   |   |
|---|---|
| a. $\cos t = \cos\left(-\frac{\pi}{4}\right)$ | b. $\cos t = -\sin\left(\frac{\pi}{8}\right)$ |
|---|---|

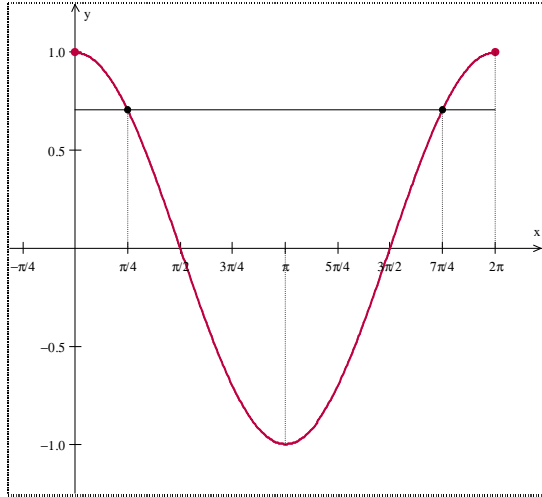
**SOLUTION:**

Draw a rough picture of the sine and cosine functions and make use of the various symmetries involved. Recall that sine is an odd function and that cosine is an even function.

a. The graph of cosine on  $[0, 2\pi]$  looks like this:



$\cos\left(-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)$ , and this value occurs twice on the graph, per the following diagram:

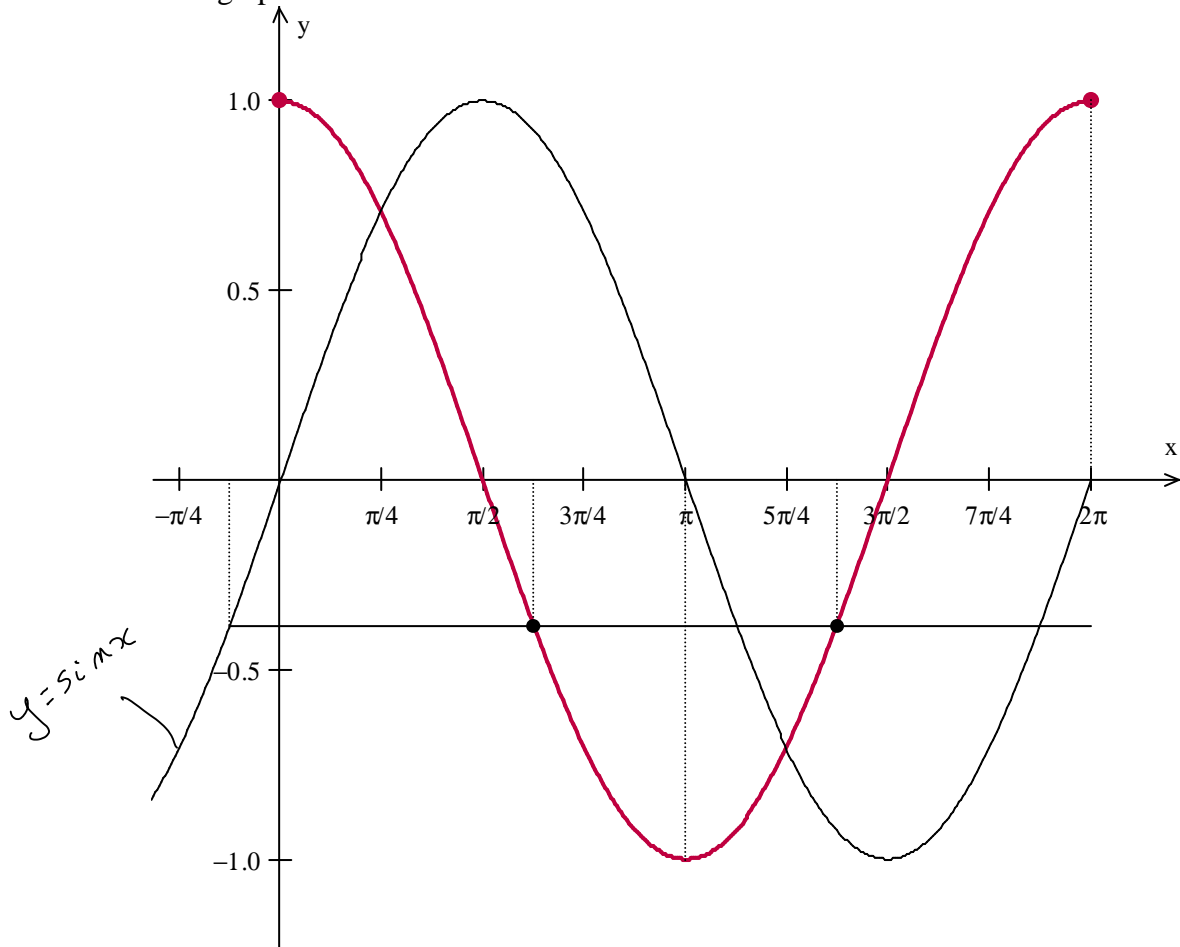


$$t = \frac{\pi}{4}, \quad t = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

The key here is that even if the value for  $\cos\left(-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)$  were unknown (which it isn't, as  $\frac{\pi}{4}$  is one of the remarkable angles, for which  $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ ), you would still be able to solve the problem, as long as the angle (in this case  $\frac{\pi}{4}$ ) can be located on the  $x$  axis. As a matter of fact, one not needs to locate the angle THAT accurately, as long as one knows roughly where the angle is located, then the various symmetries in the graph allow you to solve the problem.

- b. This next one can be solved as follows: first note that  $-\sin\left(\frac{\pi}{8}\right)=\sin\left(-\frac{\pi}{8}\right)$  because the sine function is odd. By plotting the graphs for sine and cosine simultaneously, we can then locate the position of the points on the **cosine curve** which are equal to  $-\sin\left(\frac{\pi}{8}\right)=\sin\left(-\frac{\pi}{8}\right)$ .

The graphs look like this:



Therefore:  $t = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8}, \quad t = \frac{3\pi}{2} - \frac{\pi}{8} = \frac{11\pi}{8}$

5. Show the following identity:  $\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = 2\sec^2\theta$

**SOLUTION:** develop the left side, by combining the 2 fractions, using a common denominator.

$$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = \frac{(1+\sin\theta) + (1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \frac{2}{1-\sin^2\theta} = \frac{2}{\cos^2\theta} = 2\sec^2\theta$$

6. Determine the exact value of  $\sin\left(\frac{5\pi}{12}\right)$ . HINT: rewrite  $\sin\left(\frac{5\pi}{12}\right)$  as  $\sin\left(\frac{\pi}{2} - \frac{\pi}{12}\right)$ . Then, use the half-angle formula and the subtraction identity for  $\sin(x-y)$ .

**DO NOT USE A CALCULATOR.**

**SOLUTION:**

The subtraction identity is:  $\sin(x-y) = \sin x \cos y - \sin y \cos x$

$$\text{So } \sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = \sin\frac{\pi}{2} \cos\frac{\pi}{12} - \sin\frac{\pi}{12} \cos\frac{\pi}{2} = 1 \cdot \cos\frac{\pi}{12} - \sin\frac{\pi}{12} \cdot 0 = \cos\frac{\pi}{12}$$

We can use the half-angle formula to compute  $\cos\frac{\pi}{12}$ :  $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$\text{So } \cos^2\frac{\pi}{12} = \frac{1}{2}(1 + \cos\frac{\pi}{6}) = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4}$$

Therefore  $\cos\frac{\pi}{12} = +\sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$ . We select the positive root because 'cos' is positive in the first quadrant.

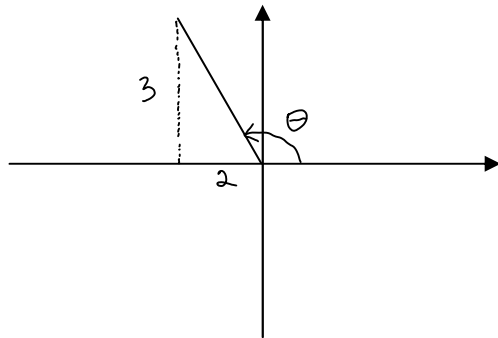
7. [3 marks] Let  $\theta$  be an angle in radians such that:  $\frac{\pi}{2} < \theta < \pi$

Determine the exact ratios  $\sin\theta$  and  $\sec\theta$  if  $\tan\theta = -\frac{3}{2}$

**SOLUTION:**

This angle is in the second quadrant, where 'sin' is positive and 'cos' is negative.

**SOLUTION 1:** we use a diagram:



And note that the hypotenuse, is  $\sqrt{(-2)^2 + 3^2} = \sqrt{13}$ , so that  $\sin\theta = \frac{\langle opp \rangle}{\langle hyp \rangle} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$

$$\text{and } \sec\theta = \frac{\langle hyp \rangle}{\langle adj \rangle} = \frac{\sqrt{13}}{-2} = -\frac{\sqrt{13}}{2}$$

**SOLUTION 2:** we use trig. identities:

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{1}{\sec^2\theta}\right)} = \sqrt{1 - \left(\frac{1}{\tan^2\theta + 1}\right)} = \sqrt{1 - \left(\frac{1}{\frac{9}{4} + 1}\right)} = \sqrt{1 - \left(\frac{4}{13}\right)} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\sec\theta = -\sqrt{\tan^2\theta + 1} = -\sqrt{\left(-\frac{3}{2}\right)^2 + 1} = -\sqrt{\frac{9}{4} + 1} = -\sqrt{\frac{13}{4}} = -\frac{\sqrt{13}}{2}$$