Name: SOLUTIONS

Marks

Unless otherwise indicated, numerical answers must be simplified completely for full marks. Show all your work. Use backs of pages if necessary. Calculators are not allowed.

[8] **1.** (a) [4] Evaluate $\int_0^1 \sqrt{x}(x+\sqrt{x})\,dx$.

$$= \int_0^1 (x^{3/2} + x)\,dx$$

$$= \left(\frac{2}{5}x^{5/2} + \frac{1}{2}x^2\right)\Big|_0^1 = \frac{2}{5} + \frac{1}{2} = \boxed{\frac{9}{10}}$$

(b) [4] Evaluate $\int \frac{(\ln x)^2}{x}\,dx$

Let $u = \ln x$, so $du = \frac{1}{x}\,dx$.

We get $\int u^2\,du = \frac{1}{3}u^3 + C$

$$= \boxed{\frac{1}{3}(\ln x)^3 + C}$$

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[12] **2.** (a) [4] Evaluate $\int (\ln x)^2\, dx$.

Use $\int$ by parts with $u = (\ln x)^2$ and $dv = dx$
$du = \frac{2\ln x}{x}\, dx$, $v = x$

We get $x(\ln x)^2 - 2\int \ln x\, dx$

Use $\int$ by parts again with $u = \ln x$, $dv = dx$
$du = \frac{dx}{x}$, $v = x$

to get $\int \ln x\, dx = x\ln x - \int dx = x\ln x - x + C$.

The final answer is $\boxed{x(\ln x)^2 - 2x\ln x + 2x + C}$

(b) [4] Evaluate $\int_{-\pi}^{\pi} x^{101} \cos x\, dx$ *without* antidifferentiation. Explain your reasoning.

Since x^{101} is an odd function and $\cos x$ is even, $x^{101}\cos x$ is odd. Thus

$$\int_{-\pi}^{\pi} x^{101} \cos x\, dx = \boxed{0}$$

(c) [4] Evaluate $\int_0^2 (x+3)(x-1)^5\, dx$.

Let $u = x - 1$ so $du = dx$ and $x = u + 1$.

$x = 0 \rightarrow u = -1$; $x = 2 \rightarrow u = 1$

Also, $x + 3 = u + 4$.

So, we have

$$\int_{-1}^{1} (u+4)u^5\, du = \int_{-1}^{1} (u^6 + 4u^5)\, du$$

$$= \int_{-1}^{1} u^6\, du + 4\int_{-1}^{1} u^5\, du$$

by symmetry $\Rightarrow = 2\int_0^1 u^6\, du + 0 = \boxed{2/7}$

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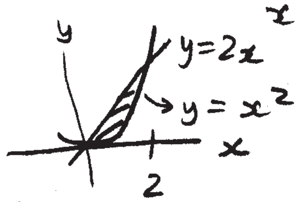
[12] **3.** (a) [4] Find the area of the region enclosed by the curves $y = x^2$ and $y = 2x$.

Intersection points: $x^2 = 2x$; $x^2 - 2x = 0$; $x(x-2) = 0$

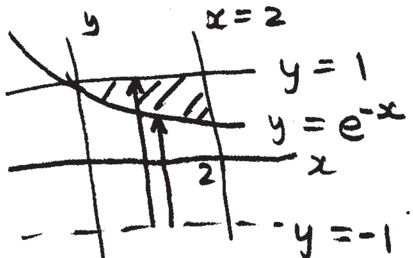
$x = 0, 2$

Sketch:



$$\text{Area} = \int_0^2 (2x - x^2)\,dx = \left(x^2 - \tfrac{1}{3}x^3\right)\Big|_0^2 = 4 - \tfrac{8}{3} = \boxed{\tfrac{4}{3}}$$

(b) [4] A solid is obtained by rotating the region enclosed by the curves $y = e^{-x}$, $y = 1$, and $x = 2$ about the line $y = -1$. Write down a definite integral giving the volume of the solid. **Do not evaluate this integral.**

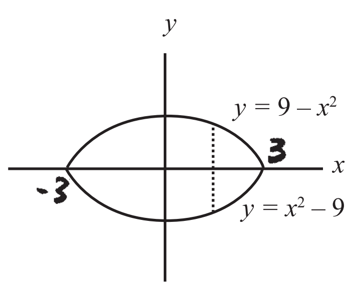


Outer radius $= 2$

Inner radius $= 1 + e^{-x}$

$$\text{Volume} = \boxed{\pi \int_0^2 \left(2^2 - (1 + e^{-x})^2\right) dx}$$

(c) [4] The horizontal base of a solid is the region enclosed by the curves $y = 9 - x^2$ and $y = x^2 - 9$, as shown in the diagram below. Each vertical cross section perpendicular to the x-axis is a triangle, whose base is drawn as a dotted line in the diagram. The height of the triangle always equals the length of the base. Write down a definite integral giving the volume of the solid. **Do not evaluate this integral.**



The dotted line, which is the base of a cross-sectional triangle, has length $2(9-x^2)$. Since the height is also $2(9-x^2)$, the area of the triangle is

$A(x) = \frac{1}{2}\left(2(9-x^2)\right)^2 = 2(9-x^2)^2$.

So, the volume is $\boxed{\int_{-3}^{3} 2(9-x^2)^2\,dx}$

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[6] **4.** Evaluate $\int_0^3 (x^2 + 4x^3)\, dx$ using a limit of Riemann sums. You may use the formulas

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

No credit will be given for a solution that uses antidifferentiation, but you may check your answer using antiderivatives.

Use right endpoints: $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $x_i = 0 + \frac{3i}{n}$

The Riemann sum is $\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{n} \left(\left(\frac{3i}{n}\right)^2 + 4\left(\frac{3i}{n}\right)^3 \right) \frac{3}{n}$$

$$= \frac{27}{n^3} \sum_{i=1}^{n} i^2 + \frac{4 \cdot 81}{n^4} \sum_{i=1}^{n} i^3$$

$$= \frac{\cancel{27}^{\,9}}{n^{\cancel{3}\,2}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,2} + \frac{\cancel{4} \cdot 81}{n^{\cancel{4}\,2}} \cdot \frac{\cancel{n^2}(n+1)^2}{\cancel{4}}$$

$$= \frac{9}{2} \frac{n+1}{n} \frac{2n+1}{n} + 81 \left(\frac{n+1}{n}\right)^2$$

$$= \frac{9}{2} \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 81 \left(1 + \frac{1}{n}\right)^2$$

Now the limit as $n \to \infty$ is $\frac{9}{2} \cdot 1 \cdot 2 + 81 \cdot 1^2$

$$= \boxed{90}$$

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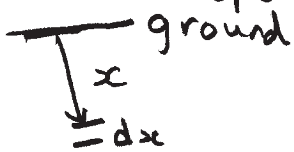
[6] **5.** A cable that has linear density 2 kg/m is attached to a bucket filled with coal that has mass 300 kg. The bucket is initially at the bottom of a 500-m-deep vertical mine shaft. The bucket is lifted to the midpoint of the shaft by winding up the top half of the cable. Find the amount of work done, in joules. You may use the value $g = 9.8$ m/s^2 for the acceleration due to gravity. *You do not need to simplify your answer in this problem, but you should evaluate any definite integral(s) that arise.*

There are <u>three</u> components:

(a) Bucket: Mass = 300 kg, Force = $300g$ N, Distance = 250m, so Work = $300 \cdot 250 g$ joules

(b) Top half of rope:



A piece of length dx at depth x has mass $2\,dx$, force $2g\,dx$, and is lifted distance x.

So, Work = $\displaystyle\int_0^{250} 2g\,x\,dx = 2g \cdot \tfrac{1}{2}x^2 \Big|_0^{250} = 250^2 g$ J

(c) Bottom half of rope: As above, but distance lifted is just 250, so

Work = $\displaystyle\int_{250}^{500} 2g \cdot 250\,dx = 2 \cdot 250^2 g$ J

Total Work = $\boxed{(300 \cdot 250 + 250^2 + 2 \cdot 250^2)\,g \text{ J}}$

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[6] **6.** (a) [3] Let $F(x) = \int_{x^2}^{7x} \frac{u}{1+u^2}\, du$. Find $F'(x)$.

$$F(x) = \int_0^{7x} \frac{u}{1+u^2}\,du - \int_0^{x^2} \frac{u}{1+u^2}\,du$$

By FTC I and the Chain Rule,

$$F'(x) = \frac{7x}{1+(7x)^2}\cdot 7 - \frac{x^2}{1+(x^2)^2}\cdot 2x$$

$$= \boxed{\frac{49x}{1+49x^2} - \frac{2x^3}{1+x^4}}$$

(b) [3] A certain continuous function $f(x)$ defined on $[0,\infty)$ has the property that the average value of $f(x)$ on the interval $[0, x^2]$ equals x, for all $x > 0$. Determine $f(x)$, with explanation.

We have $\frac{1}{x^2-0}\int_0^{x^2} f(t)\,dt = x$

$$\int_0^{x^2} f(t)\,dt = x^3$$

Using FTC I and Chain Rule, differentiating gives $2x\,f(x^2) = 3x^2$

$$f(x^2) = \frac{3}{2}x$$

$$f(x) = f((\sqrt{x})^2) = \boxed{\frac{3}{2}\sqrt{x}}$$

The End