MAT 1302E, Fall 2012

Assignment 1

Professor: Eric Hua

Due date: September 25, 2012, by 5:30p.m., in class. Total points: 13

For all of the questions below, you must show each step in any row reduction and state what operation you are performing at each step.

Do NOT leave submission of your assignment until the last minute. Late assignments will NOT be accepted.

Grading Scheme: Please be somewhat lenient with respect to formatting. For instance, don't take off marks for missing bars in augmented matrices or systems of equations that aren't aligned perfectly.

1. **(4 points)** Solve the following system using row reduction. Check your answer.

$$\begin{array}{rcrcrcr} -x_1 & + & 3x_2 & + & 2x_3 & = & 2 \\ x_1 & - & 2x_2 & - & 2x_3 & = & -\frac{7}{3} \\ x_1 & - & 3x_2 & - & x_3 & = & -1 \end{array}$$

Solution: We start with the augmented matrix and row reduce it to obtain the *reduced echelon form.*

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 2 \\ 1 & -2 & -2 & -\frac{7}{3} \\ 1 & -3 & -1 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} -1 & 3 & 2 & 2 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{ccc|c} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to -R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Switching back to equation notation gives

$$x_1 = -1$$
$$x_2 = -\frac{1}{3}$$
$$x_3 = 1$$

and so the system has the unique solution $(-1, -\frac{1}{3}, 1)$. To check our answer, we substitute this solution back into the original equations:

$$\begin{array}{rcrcrcr} -(-1) & + & 3(-\frac{1}{3}) & + & 2(1) & = & 2 \\ (-1) & - & 2(-\frac{1}{3}) & - & 2(1) & = & -\frac{7}{3} \\ (-1) & - & 3(-\frac{1}{3}) & - & 1 & = & -1 \end{array}$$

Since each substitution yields a true statement, we have verified that $(-1, -\frac{1}{3}, 1)$ is indeed a solution.

Grading Scheme:

- 2 points for the correct row reduction (0, 1, or 2).
- 1 point for writing down the correct solution from their RREF.
- 1 point for checking the answer.

2. **(4 points)** The following matrix is the augmented matrix of a linear system.

$$\left[\begin{array}{cccc|c} -2 & 0 & 3 & 1 & -1 \\ 0 & -1 & 1 & 1 & 2 \\ 2 & 0 & -1 & -4 & 1 \\ -2 & -1 & 4 & 2 & 0 \end{array}\right]$$

Determine the pivot columns of the above matrix. Determine if the linear system is consistent or inconsistent. You should justify your answers. (You do not need to completely solve the linear system if it is consistent.)

Solution: We row reduce to obtain an echelon form.

$$\left[\begin{array}{cccc|c} -2 & 0 & 3 & 1 & -1 \\ 0 & -1 & 1 & 1 & 2 \\ 2 & 0 & -1 & -4 & 1 \\ -2 & -1 & 4 & 2 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_1]{R_3 \to R_3 + R_1} \left[\begin{array}{cccc|c} -2 & 0 & 3 & 1 & -1 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & -1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_4 \to R_4 - R_2} \left[\begin{array}{cccc|c} -2 & 0 & 3 & 1 & -1 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

The pivot columns are the first, second, third, and fifth columns. Since the rightmost column is a pivot column, the system is inconsistent.

Grading Scheme:

- 2 points for correct row reduction (0, 1, or 2).
- 1 point for the correct list of pivot columns (0 or 1).
- 1 point for stating the system is inconsistent, with justification (0 or 1).

3. **(5 points)** Find the general solution of the following system. Indicate which variables are basic and which are free. Check your final answer.

$$\begin{aligned} x_1 + 4x_4 + 3 &= x_2 + x_3 \\ x_1 + 3x_4 + 1 &= \frac{1}{2}x_3 \\ x_1 + x_2 + 2x_4 &= 1 \end{aligned}$$

Solution: We find the augmented matrix and row reduce.

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 4 & -3 \\ 1 & 0 & -\frac{1}{2} & 3 & -1 \\ 1 & 1 & 0 & 2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 4 & -3 \\ 0 & 1 & \frac{1}{2} & -1 & 2 \\ 0 & 2 & 1 & -2 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 4 & -3 \\ 0 & 1 & \frac{1}{2} & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & 3 & -1 \\ 0 & 1 & \frac{1}{2} & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The variables x_1 and x_2 correspond to pivot columns, hence they are *basic*. The variables x_3 and x_4 are *free*. Solving for the basic variables in terms of the free ones, we obtain the general solution of the system:

$$\begin{aligned} x_1 &= -1 + \frac{1}{2}x_3 - 3x_4 \\ x_2 &= 2 - \frac{1}{2}x_3 + x_4 \\ x_3 &= \text{free} \\ x_4 &= \text{free.} \end{aligned}$$

We check our answer by substituting our expressions for x_1, x_2 into the original equations (leaving in the free variables x_3, x_4).

$$\begin{aligned} (-1+\tfrac{1}{2}x_3-3x_4)+4x_4+3 &= 2-\tfrac{1}{2}x_3+x_4+x_3 \\ (-1+\tfrac{1}{2}x_3-3x_4)+3x_4+1 &= \tfrac{1}{2}x_3 \\ (-1+\tfrac{1}{2}x_3-3x_4)+(2-\tfrac{1}{2}x_3+x_4)+2x_4 &= 1 \end{aligned}$$

Since each substitution yields a true statement, we have verified our solution.

Grading Scheme:

- 2 points for the correct row reduction (0, 1, or 2).
- 2 points for the correct general solution (0, 1, or 2).
- 1 point for checking the answer (give the student 1 point if their solution is not correct but their method of checking it based on the incorrect solution is still correct).