School of Mathematics and Statistics
Carleton University
Math. 1004A, Fall 2011

TEST 1

SOLUTIONS

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[5 marks]** Let $f(x) = \sqrt{x} + 1/x$. Then $f'(4)$ is equal to:

   (a) 0, (b) 1/4, (c) 1/2, (d) 3/16. Answer: (d), a simple calculation.

2. **[5 marks]** Let f be defined by

$$f(t) = \begin{cases} t-1, & \text{if} \quad t \leq 0, \\ \dfrac{1-\cos t}{t}, & \text{if} \quad t > 0. \end{cases}$$

   Is f continuous at $t = 0$?

   (a) YES, (b) NO, Answer: (b), since the right- and left-hand limits at 0 are different.

3. **[5 marks]** Evaluate the limit: $\lim\limits_{u \to 0} \dfrac{\sin 3u}{\sin 2u}$.

   (a) 1/2 , (b) −1 , (c) 3/2, (d) 0.

   Answer: (c), since $\dfrac{\sin 3u}{\sin 2u} = \dfrac{\sin 3u}{3u}\dfrac{2u}{\sin 2u}\dfrac{3u}{2u}$. Now cancel the last two u's and let $u \to 0$ using a basic limit theorem about the sine function.

4. **[5 marks]** Define two functions f, g be defined by $f(x) = \dfrac{x-1}{x+1}$ and $g(t) = 1 + \sin t$. Find the derivative of the composition $(f \cdot g)(t)$ at the point $t = 0$.

   (a) 1/2, (b) 1, (c) 2, (d) 0.

   Answer: (a), since, by the Chain Rule, $D(f(g(t)) = f'(g(t))g'(t)$. But $f'(x) = 2/(x+1)^2$, $g'(t) = \cos t$. Therefore, $f'(g(t)) = 2/(g(t)+1)^2$ and the result follows by the Chain Rule upon letting $t = 0$.

5. **[5 marks]** Answer TRUE or FALSE: The function f defined by

$$f(t) = \begin{cases} t, & \text{if} \quad 0 \leq t \leq 1, \\ t^2, & \text{if} \quad -1 \leq t < 0, \end{cases}$$

   is one-to-one on the interval $[-1, 1]$.

   (a) TRUE, (b) FALSE Answer: (b), as the graph of f obviously fails the horizontal line test.

PART II: Show all work here.
No additional pages will be accepted

6. **[10+ 8 marks]** :

   a) $f(t) = \sin(\tan\,(3t+1))$. Find the derivative, $f'(t)$ (there is no need to simplify your answer).

   b) Evaluate the following limit: $\lim\limits_{x \to \infty} x \sin(1/x)$.

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a) $D \sin(\square) = \cos \square \cdot D\square$ $\therefore$ if $\square = \tan(3t+1)$

$D\square = \sec^2(3t+1) \cdot 3$ ← 2

$\therefore f'(t) = \underbrace{\cos(\tan(3t+1))}_{2} \cdot \underbrace{\sec^2(3t+1)}_{2} \cdot \underbrace{3}_{2}$

Write $x \sin\frac{1}{x} = \dfrac{\sin\frac{1}{x}}{\frac{1}{x}}$. ← ③

① ↘ ↙ ①

As $x \to \infty$, $\frac{1}{x} \to 0$. $\therefore$ $\square = \frac{1}{x}$ gives $\square \to 0$ so that

$$\lim_{x \to \infty} x \sin\frac{1}{x} = \lim_{\square \to 0} \left(\frac{\sin \square}{\square}\right) \quad \leftarrow ②$$

$$= 1. \quad \leftarrow ①$$

7. **[8+9 marks]**

a) Evaluate the following one-sided limit: $\displaystyle\lim_{x\to 1^+} \frac{1-\cos(x-1)}{\sqrt{x-1}}$.

b) A differentiable function f has the property that $f(\pi) = -1$, $f'(-1) = 4$ and $f'(\pi) = 2$. What is the value of the derivative of $f(f(x))$ at $x = \pi$?

a) Write $\dfrac{1-\cos(x-1)}{\sqrt{x-1}} = \dfrac{1-\cos(x-1)}{(x-1)} \cdot \sqrt{x-1}$ ← ②

↙ ①

As $x \to 1^+$, $\dfrac{1-\cos(x-1)}{(x-1)} \to 0$ ↖ ① $\quad \left(\square = x-1 \to 0 \text{ as } x \to 1^+ \;\; \therefore \dfrac{1-\cos\square}{\square} \to 0 \right)$.

↗ ①

① → & $\sqrt{x-1} \to 0$ too.

$$\therefore \lim_{x\to 1^+} \frac{1-\cos(x-1)}{\sqrt{x-1}} = 0.$$

↑ ②

↙ ③

b) Chain Rule says $Df(f(x)) = f'(f(x))f'(x)$. So at $x = \pi$,

$$\therefore f'(f(\pi))f'(\pi) = f'(-1)f'(\pi) \quad \leftarrow ③$$
$$= 4 \cdot 2 \quad \leftarrow ②$$
$$= 8 \quad \leftarrow ①$$
