

MATH 3705 Final Examination Solutions
April 2004

1. $\mathcal{L}\{e^{2t} \cos(3t)\} =$ (a)

(a) $\frac{s - 2}{(s - 2)^2 + 9}$

(b) $\frac{s}{(s - 2)^2 + 9}$

(c) $\frac{s - 2}{s^2 + 9}$

(d) $\frac{e^{-2s}}{s^2 + 9}$

(e) None of the above.

2. $\mathcal{L}\{t \sin(2t)\} =$ (d)

(a) $\frac{s}{(s^2 + 4)^2}$

(b) $\frac{-s}{(s^2 + 4)^2}$

(c) $\frac{2}{(s - 1)^2 + 4}$

(d) $\frac{4s}{(s^2 + 4)^2}$

(e) None of the above.

3. $\mathcal{L}^{-1}\left\{\frac{3e^{-2s}}{s^2 + s - 2}\right\} =$ (b)

(a) $u(t - 2)[e^t - e^{-2t}]$

(b) $u(t - 2)[e^{t-2} - e^{-2t+4}]$

(c) $e^{t-2} - e^{-2(t-2)}$

(d) $u(t)e^t - u(t - 2)e^{t-2}$

(e) None of the above.

4. $\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2s + 10} \right\} =$ (d)

(a) $e^{-t} \cos(3t)$

(b) $e^{-t} \sin(3t)$

(c) $e^{-t} \cos(3t) - e^{-t} \sin(3t)$

(d) $e^{-t} [\cos(3t) - \frac{1}{3} \sin(3t)]$

(e) None of the above.

5. The general solution of the differential equation $2x^2y'' - 5xy' + 3y = 0$, valid for $x \neq 0$, is given by (a)

(a) $c_1|x|^3 + c_2|x|^{\frac{1}{2}}$

(b) $|x|^3[c_1 + c_2 \ln |x|]$

(c) $|x|^3 \left[c_1 \cos \left(\frac{1}{2} \ln |x| \right) + c_2 \sin \left(\frac{1}{2} \ln |x| \right) \right]$

(d) $c_1|x|^{\frac{1}{2}} + c_2|x|^3 \ln |x|$

(e) None of the above.

6. The general solution of the differential equation $x^2y'' + 2xy' + \frac{1}{4}y = 0$, valid for $x \neq 0$, is given by (d)

(a) $|x|^{-1} \left[c_1 \cos \left(\frac{\sqrt{3}}{2} \ln |x| \right) + c_2 \sin \left(\frac{\sqrt{3}}{2} \ln |x| \right) \right]$

(b) $c_1|x|^{-1} + c_2|x|^{\frac{\sqrt{3}}{2}}$

(c) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{-\frac{1}{2}}$

(d) $|x|^{-\frac{1}{2}}(c_1 + c_2 \ln |x|)$

(e) None of the above.

7. The general solution of the differential equation $x^2y'' + xy' + (7x^2 - 4)y = 0$, valid for $x > 0$, is given by (c)

(a) $c_1J_2(\sqrt{7}x) + c_2J_{-2}(\sqrt{7}x)$

(b) $c_1J_{\sqrt{7}}(2x) + c_2J_{-\sqrt{7}}(2x)$

(c) $c_1J_2(\sqrt{7}x) + c_2Y_2(\sqrt{7}x)$

(d) $c_1J_{\sqrt{7}}(2x) + c_2Y_{\sqrt{7}}(2x)$

(e) None of the above.

8. Let $f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases}$ on $[0, 2]$. At $x = 79$, the Fourier sine series of f converges to (d)

(a) 2

(b) $\frac{3}{2}$

(c) 0

(d) $-\frac{3}{2}$

(e) None of the above.

9. The differential equation $y'' - 2y' + xy + \lambda y = 0$, when placed in the Sturm-Liouville form $(py')' - qy + \lambda ry = 0$, has the weight function $r(x) =$ (c)

(a) $-xe^{-2x}$

(b) xe^{-2x}

(c) e^{-2x}

(d) x

(e) None of the above.

10. Given the Bessel identity $\frac{1}{\alpha} \frac{d}{dx} [x^\nu J_\nu(\alpha x)] = x^\nu J_{\nu-1}(\alpha x)$, $\nu > 0$, $\alpha \neq 0$, $\int_0^3 x^4 J_1(2x) dx =$ (a)

(a) $\frac{1}{2}[81J_2(6) - 27J_3(6)]$

(b) $81J_2(6) - 27J_3(6)$

(c) $27J_3(6) - 81J_2(6)$

(d) $\frac{243}{5}J_2(6)$

(e) None of the above.

11. $\mathcal{F}\{e^{-2ix-|x+3|}\} =$ (c)

(a) $\frac{2e^{-3i\lambda}}{1 + (\lambda - 2)^2}$

(b) $\frac{2e^{-3i(\lambda-2)}}{1 + \lambda^2}$

(c) $\frac{2e^{-3i(\lambda-2)}}{1 + (\lambda - 2)^2}$

(d) $\frac{2e^{-3i\lambda}}{1 + (\lambda + 2)^2}$

(e) None of the above.

12. $\mathcal{F}\{xe^{-3x^2}\} =$ (b)

(a) $\frac{\sqrt{\pi}}{3}\lambda e^{-\frac{\lambda^2}{12}}$

(b) $\frac{i\sqrt{\pi}}{6\sqrt{3}}\lambda e^{-\frac{\lambda^2}{12}}$

(c) $\frac{\sqrt{\pi}}{3}e^{-3\lambda^2}$

(d) $\frac{1}{2\pi}\lambda e^{-3\lambda^2}$

(e) None of the above.

13. $\mathcal{F}^{-1} \left\{ \frac{e^{2i\lambda}}{1 + (\lambda + 3)^2} \right\} =$ (d)

(a) $\frac{1}{2}e^{3ix - |x-2|}$

(b) $\frac{1}{2}e^{3i(x-2) - |x|}$

(c) $\frac{1}{2}e^{3i(x+2) - |x+2|}$

(d) $\frac{1}{2}e^{3i(x-2) - |x-2|}$

(e) None of the above.

14. $\mathcal{F}^{-1}\{\lambda e^{-|\lambda|}\} =$ (b)

(a) $\frac{2ix}{\pi(1+x^2)^2}$

(b) $\frac{-2ix}{\pi(1+x^2)^2}$

(c) $\frac{x}{\pi(1+x^2)}$

(d) $\frac{-x}{\pi(1+x^2)}$

(e) None of the above.

15. Employ the Laplace transform to solve the initial-value problem

$$y'' - 4y' + 13y = \delta(t - 3), \quad y(0) = 1, \quad y'(0) = 5.$$

$$s^2Y(s) - sy(0) - y'(0) - 4[sY(s) - y(0)] + 13Y(s) = e^{-3s} \Rightarrow (s^2 - 4s + 13)Y(s) - s - 1 = e^{-3s}$$

$$\Rightarrow Y(s) = \frac{s+1}{s^2 - 4s + 13} + \frac{e^{-3s}}{s^2 - 4s + 13} = \frac{(s-2)+3}{(s-2)^2 + 9} + \frac{e^{-3s}}{(s-2)^2 + 9}$$

$$\Rightarrow y(t) = e^{2t} \cos(3t) + e^{2t} \sin(3t) + \frac{1}{3}u(t-3)e^{2(t-3)} \sin[3(t-3)].$$

16. Find one (non-zero) series solution y_1 of the differential equation $xy'' + 2xy' + 2y = 0$, valid for $x > 0$ near $x_0 = 0$. Express the solution as an elementary function.

$$xp(x) = 2x, p_0 = 0, x^2q(x) = 2x, q_0 = 0 \Rightarrow r^2 + (p_0 - 1)r + q_0 = r^2 - r = r(r - 1) = 0 \Rightarrow r_1 = 1$$

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{n+1}, y' = \sum_{n=0}^{\infty} (n+1)a_n x^n, y'' = \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n+1)a_n x^n + \sum_{n=0}^{\infty} 2(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+1} + [2(n+1) + 2]a_n = 0, n \geq 0, \Rightarrow a_{n+1} = -\frac{2a_n}{n+1}, n \geq 0.$$

$$n = 0 \Rightarrow a_1 = -\frac{2a_0}{1}, n = 1 \Rightarrow a_2 = -\frac{2a_1}{2} = \frac{2^2 a_0}{1 \cdot 2},$$

$$n = 2 \Rightarrow a_3 = -\frac{2a_2}{3} = -\frac{2^3 a_0}{1 \cdot 2 \cdot 3}, \text{ etc., so}$$

$$a_n = \frac{(-1)^n 2^n a_0}{n!}, n \geq 0, \Rightarrow y = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n a_0}{n!} x^{n+1} = a_0 x \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = a_0 x e^{-2x}.$$

17. Find the Fourier cosine series of $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & 1 \leq x \leq 2 \end{cases}$ on $[0, 2]$.

The series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$, with

$$a_0 = \frac{2}{2} \int_0^2 f(x) dx = \int_0^1 x dx = \frac{1}{2}, \text{ and for } n \geq 1,$$

$$a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^1 x \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2}{n\pi} x \sin\left(\frac{n\pi x}{2}\right) \Big|_0^1 - \frac{2}{n\pi} \int_0^1 \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2 \pi^2} \cos\left(\frac{n\pi x}{2}\right) \Big|_0^1$$

$$= \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2 \pi^2} \left[\cos\left(\frac{n\pi}{2}\right) - 1 \right]$$

Thus, the cosine series of f is

$$\frac{1}{4} + \sum_{n=1}^{\infty} \left\{ \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2 \pi^2} \left[\cos\left(\frac{n\pi}{2}\right) - 1 \right] \right\} \cos\left(\frac{n\pi x}{2}\right).$$

18. The solution of the heat equation $u_{xx} = \frac{1}{\alpha^2}u_t$, $0 < x < L$, which satisfies the boundary conditions $u_x(0, t) = u_x(L, t) = 0$, has the form

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution of $u_{xx} = \frac{1}{9}u_t$, $0 < x < \pi$, which satisfies the boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$ and the initial condition $u(x, 0) = 3 \cos(2x) - 2 \cos(5x)$. Write down the complete solution $u(x, t)$.

$$\alpha = 3, L = \pi \Rightarrow u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-9n^2 t}.$$

$$3 \cos(2x) - 2 \cos(5x) = u(x, 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \Rightarrow a_2 = 3, a_5 = -2, a_n = 0 \text{ otherwise.}$$

$$\text{Thus, } u(x, t) = 3 \cos(2x) e^{-36t} - 2 \cos(5x) e^{-225t}.$$

19. The solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ inside the circle $r = a$ has the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Find the solution of Laplace's equation inside the circle $r = 2$, which satisfies the boundary condition $u(2, \theta) = 2 + \sin(2\theta) - \cos(\theta)$. Write down the complete solution $u(r, \theta)$.

$$2 + \sin(2\theta) - \cos(\theta) = u(2, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

$$\Rightarrow \frac{a_0}{2} = 2, 2a_1 = -1, 2^2 b_2 = 1, a_n = b_n = 0 \text{ otherwise. Thus,}$$

$$a_0 = 4, a_1 = -\frac{1}{2}, b_2 = \frac{1}{4} \Rightarrow u(r, \theta) = 2 - \frac{1}{2}r \cos(\theta) + \frac{1}{4}r^2 \sin(2\theta).$$

20. Find all eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$.

$$y = e^{rx} \Rightarrow r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}.$$

$$\begin{aligned} \lambda = -\mu^2 < 0 \Rightarrow r = \pm\mu \Rightarrow y = Ae^{\mu x} + Be^{-\mu x}, y(0) = 0 \Rightarrow A + B = 0 \\ \Rightarrow y = A(e^{\mu x} - e^{-\mu x}), y' = A\mu(e^{\mu x} + e^{-\mu x}), y'(1) = 0 \Rightarrow A\mu(e^\mu + e^{-\mu}) = 0 \\ \Rightarrow A = 0 \Rightarrow B = 0 \Rightarrow y \equiv 0. \end{aligned}$$

$$\lambda = 0 \Rightarrow y = Ax + B, y(0) = 0 \Rightarrow B = 0 \Rightarrow y = Ax, y' = A, y'(1) = 0 \Rightarrow A = 0 \Rightarrow y \equiv 0.$$

$$\begin{aligned} \lambda = \mu^2 > 0 \Rightarrow y = A \cos(\mu x) + B \sin(\mu x), y(0) = 0 \Rightarrow A = 0 \Rightarrow y = B \sin(\mu x), \\ y' = B\mu \cos(\mu x), y'(1) = 0 \Rightarrow B\mu \cos(\mu) = 0 \Rightarrow \mu_n = (2n + 1)\frac{\pi}{2}, n \geq 0 \end{aligned}$$

$$\Rightarrow \lambda_n = \frac{(2n + 1)^2 \pi^2}{4}, y_n = B_n \sin \left[\frac{(2n + 1)\pi x}{2} \right], n \geq 0.$$