University of Ottawa
MAT 1330B Midterm Exam

October 19, 2011, Duration: 80 minutes. Instructor: Jason Levy

Family Name: ____

First Name: ____

DGD 1 DGD 2 DGD 3 DGD 4

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Ta4e your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Ma4e a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any 4ind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved TI-30 calculator is allowed.
- The correct answer requires justification written legibly and logically: you must convince me that you 4now why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good Luck!

Student number:____________________, Total mar4s: ___________ out of 30

Problem	1	2	3	4	5	6	7
Mar4s							

Question 1. Use the definition of derivative to calculate the derivative of

$$f(x) = 4x + 3.$$

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(4(x+h)+3) - (4x+3)}{h} \\ &= \lim_{h\to 0} \frac{4x + 4h + 3 - 4x - 3}{h} = \lim_{h\to 0} \frac{4h}{h} = \lim_{h\to 0} 4 = 4. \end{aligned}$$

derivative = 4

Question 2. (a) Estimate the limit

$$\lim_{x\to 0^+} \frac{\sqrt{2-2\cos(4x)}}{x}$$

using a sequence. Two terms are sufficient. (Hint: make sure your calculator is set to radians!)

Solution: We evaluate at some small positive values of x. For convenience, we consider $x = 0.1$ and $x = 0.01$. To three decimal places, we have

$$\frac{\sqrt{2-2\cos(4(0.1))}}{0.1} = 3.973$$
$$\frac{\sqrt{2-2\cos(4(0.01))}}{0.01} = 4.000$$

We guess that the limit will be 4.

Estimate for (a): 4

(b) Estimate the limit

$$\lim_{x\to 0^-} \frac{\sqrt{2-2\cos(4x)}}{x}$$

using a sequence. One term is sufficient.

Solution: We evaluate at a small negative value of x. For convenience, we consider $x = -0.01$. To three decimal places, we have

$$\frac{\sqrt{2-2\cos(4(-0.01))}}{-0.01} = -4.000$$

Estimate for (b): -4

(c) Does the limit

$$\lim_{x\to 0} \frac{\sqrt{2-2\cos(4x)}}{x}$$

exist? No Briefly explain your reasoning.

Solution: The limit only exists if the left and right limits are equal. Since they are not equal, the limit will not exist.

Question 3. (a) What is the domain of

$$f(x) = \frac{e^{5x} + x^4}{4x + 1}?$$

Solution: The domain is $\{x \mid x \neq -1/4\}$.

(b) Calculate the derivative of this function. (Do not simplify your result.)

Solution: We use the quotient rule and the chain rule. Notice that

$$\frac{d}{dx}\left(e^{5x} + x^4\right) = 5e^{5x} + 4x^{4-1}$$

and

$$\frac{d}{dx}(4x + 1) = 4,$$

and the quotient rule states that

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

We end up with:

$$f'(x) = \boxed{\frac{\left(5e^{5x} + 4x^{4-1}\right)(4x + 1) - \left(e^{5x} + x^4\right)(4)}{(4x + 1)^2}}$$

Question 4. Evaluate the limit

$$\lim_{x \to 4} \frac{\sin(x^2) + \cos^2(2x) + e^x}{x^2 - 3x + 5}.$$

You do **not** need to simplify your answer. You may use any results from class.

$$\text{limit} = \boxed{\frac{\sin(4^2) + \cos^2(8) + e^4}{4^2 - 12 + 5}}$$

Justify your answer in a short sentence.

Solution: Both the numerator and denominator are continuous and defined at $x = 4$, and the denominator is nonzero at $x = 4$, so the quotient is also continuous at $x = 4$. That means that the value of the limit is obtained simply by substituing 4 for x.

Question 5. Evaluate the following limit exactly, showing your work:

$$\lim_{h\to 0} \frac{(4+h)^2 - 4^2}{\sqrt{4+h} - \sqrt{4}}$$

Solution: We need to cross-multiply by the sums of the square roots. This gives

$$\lim_{h\to 0} \frac{(4+h)^2 - 4^2}{\sqrt{4+h} - \sqrt{4}} = \lim_{h\to 0} \left[\frac{(4+h)^2 - 4^2}{\sqrt{4+h} - \sqrt{4}}\right] \left[\frac{\sqrt{4+h} + \sqrt{4}}{\sqrt{4+h} + \sqrt{4}}\right]$$

$$= \lim_{h\to 0} \frac{((4+h)^2 - 4^2)\,(\sqrt{4+h} + \sqrt{4})}{4+h-4}.$$

The denominator simplifies to h. Now we expand out $(4+h)^2$ to get

$$= \lim_{h\to 0} \frac{(4^2 + 8h + h^2 - 4^2)\,(\sqrt{4+h} + \sqrt{4})}{h}$$

$$= \lim_{h\to 0} \frac{(8h + h^2)\,(\sqrt{4+h} + \sqrt{4})}{h}$$

$$= \lim_{h\to 0} (8+h)\,(\sqrt{4+h} + \sqrt{4}),$$

where we cancelled out a factor of h in the numerator and denominator. We are left with a function that is continuous at $h = 0$ (the expression is no longer of the form $0/0$), so we can evaluate the limit simply by plugging in $h = 0$. This gives

$$= 8(\sqrt{4} + \sqrt{4}) = 16\sqrt{4} = 32.$$

limit = 32

Question 6. [8 points] Consider the discrete-time dynamical system (DTDS)

$$M_{t+1} = -0.75M_t + 3$$

(a) [1 point] Find the updating function of the DTDS.

$f(x) = -0.75x + 3$

(b) [1 point] Find the equilibrium point of the DTDS.

$\frac{3}{1.75} = \frac{12}{7} \sim 1.71$

(c) [2 points] Give the general solution formula for the DTDS:

$M_t =$

$M_t = \frac{12}{7} + (-0.75)^t(M_0 - \frac{12}{7})$

(d) [1 point] Calculate M_{10} if $M_0 = 0$. $M_{10}=$

1.618

(e) [2 points] Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.

(f) [1 point] Is the equilibrium point stable or unstable?

stable

Question 7. (a) Find the critical point(s) of the function $f(x) = (-x^2 + 2x + 2)e^{-x}$.

Solution: We find $f'(x)$ using the product rule

$$f'(x) = (-2x + 2)e^{-x} + (-x^2 + 2x + 2)(-e^{-x}) = (x^2 - 4x)e^{-x} = x(x - 4)e^{-x}$$

This is defined everywhere, and equals 0 only when $x = 0$ or $x = 4$.

critical points: $0, 4$

(b) Find the intervals where the function is increasing and where it is decreasing.

Solution: When $f'(x)$ is positive, the function $f(x)$ is increasing, and when $f'(x)$ is negative, the function $f(x)$ is decreasing. Since e^{-x} is always positive, we get the answer

increasing: $x > 4$ or $x < 0$

decreasing: $0 < x < 4$