Continuous Probability Models

Week 5b

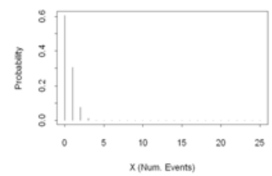
Prof. Brand

ADM2303

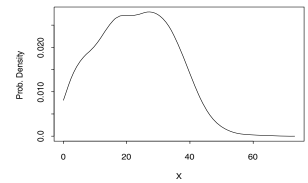
February 7, 2012

Definition, notation, and concept

Discrete
Probability Mass Function (PMF)



Continuous
Probability Density Function (PDF)



Definition, notation, and concept

Discrete

$$E[X] = \sum_{i=1}^{n} x_i \Pr[x_i]$$

$$Var[X] = \sum_{i=1}^{n} (x_i - \mu)^2 \Pr[x_i]$$

Prob. elementary $\Pr[X = x] = ?$

Prob. Event $\Pr[X > a] = \sum_{x>a} \Pr[X = x]$

Continuous

$$E[X] = \int_a^b x f_x(x)$$

$$Var[X] = \int_a^b (x-\mu)^2 f_x(x)$$

Prob. elementary $\Pr[X = x] = 0$

Prob. Event $\Pr[X > a] = \int_{x>a} (f_x(x)dx)$

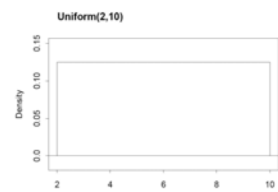
Uniform Probability Density Function (PDF)

A uniform probability density function has two parameters (a and b) and can be expressed as

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } b < x < a \\ 0 & \text{otherwise} \end{cases}$$

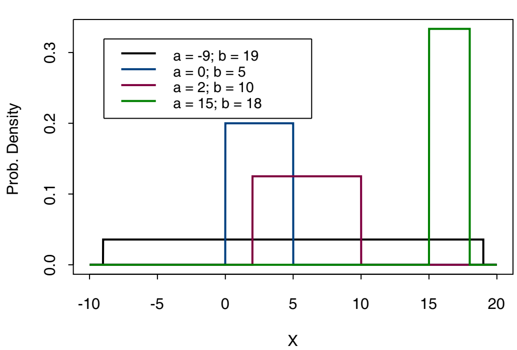
mean $\mu_x = \frac{a+b}{2}$

variance $\sigma_x^2 = \frac{(b-a)^2}{12}$



where a and b represent the bounds (minimum and maximum, respectively) between which x has nonzero probability density.

Family of Uniform PDF's



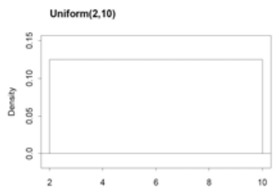
Example application of Uniform PDF

If X is uniformly distributed over (2,10) calculate the probability that:

(a) $X < 3$,
(b) $X > 6$,
(c) $3 < X < 6$,
(d) $X < 1$,
(e) its expected value (μ_x), and
(f) its variance (σ_x^2)?

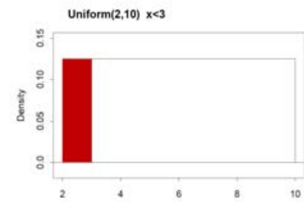
Solution to Area Questions

- Identify parameters of uniform distribution ($a = 2, b = 10$),
- Draw the density function (PDF). Note breadth of 'support' is $b - a = 8$.



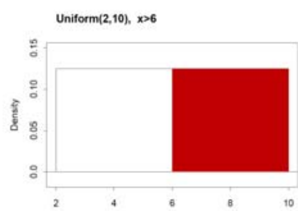
(a) X<3

(a) $P[X < 3]$ Note that $(x-a) = (3-2) = 1$ constitutes $1/8$ of the total support and thus $1/8$th of the area.

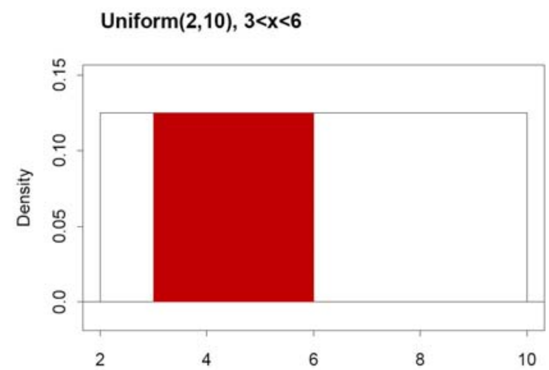


(b) X>6

(b) $P[X > 6]$ Note that $(b - x) = (10 - 6) = 4$ constitutes $4/8$ of the total support and thus $4/8$th of area.



(c) 3<X<6



For remaining questions refer to your notes (expressions for mean and variance) for the uniform PDF.

mean

$$\begin{aligned}\mu_x &= \frac{(b+a)}{2} \\ &= \frac{(10+2)}{2} = 6\end{aligned}$$

variance

$$\begin{aligned}\sigma_x^2 &= \frac{(b-a)^2}{12} \\ &= \frac{(10-2)^2}{12} \\ &= \frac{64}{12} = 16/3\end{aligned}$$

Normal PDF

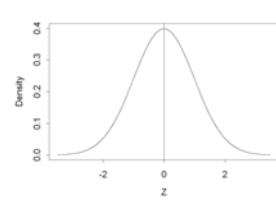
A normal probability density function has two parameters. Conveniently these are its mean μ_x and its standard deviation σ_x.
The normal or *Gaussian* probability density function is expressed as,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)$$

bounds $-\infty < x < \infty$

mean $\mu_x = \mu_x$

variance $\sigma_x^2 = \sigma_x^2$



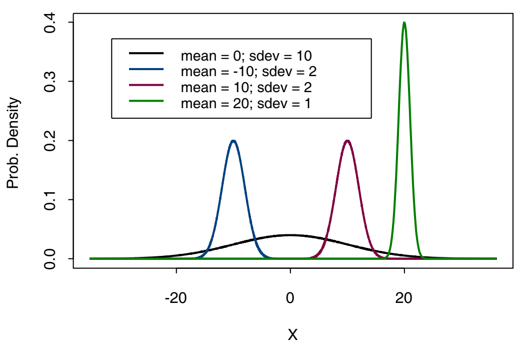
Normal Distribution

- Perhaps most widely known probability model
- Widely applicable (to nature, physiology, intelligence scores, and etc)
- Many variables in business are normally distributed,
- Sometimes referred to as a Gaussian distribution or the "Bell (shaped)" curve

Normal PDF: Properties

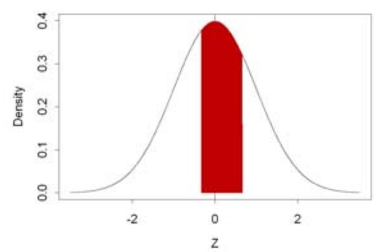
- Symmetric
- Unbounded (bounds $X \in [-\infty, +\infty]$)
- Only one mode (peak)
- Bell-shaped
- Really represents a family of curves

Family of Normal PDFs



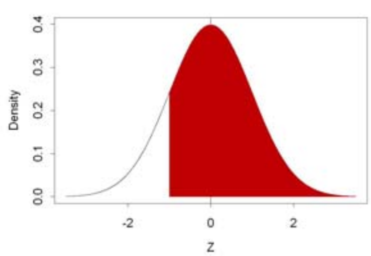
Basic example (1)

1. If X is a normal random variable with parameters $\mu_x = 3$ and $\sigma_x = 3$, find:
   (a) $P[2 < X < 5]$,



Basic example (2)

1. X is a normal RV with parameters $\mu_x = 3$ and $\sigma_x = 3$, find:
   (b) $P[X > 0]$.



Gauging Area: The 68/95/99 Rule

For any normal model

- Roughly **68 percent** of probability is bracketted between $\pm 1\sigma$ of its μ;
- Roughly **95 percent** of probability is bracketted between $\pm 2\sigma$ of its μ;
- Roughly **99 percent** of probability is bracketted between $\pm 3\sigma$ of its μ.

Gauging Area: The 68/95/99 Rule

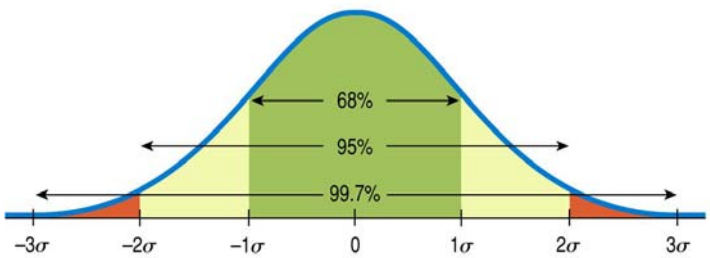


Figure: Reproduced from Sharpe et al, 2009

Standardized Normal

- Since the number of possible μ's and σ's (each defining a unique normal model) is unlimited, there are an unlimited number of unique normal PDF.
- We could tabulate "area's" for a single normal model for various values of X, but we could not do so for all models,
- Why not tabulate "area's" for a standard normal PDF, and then use the standard results to address the general case,
- Thus, the motivation behind the **"Standard Normal Distribution"**

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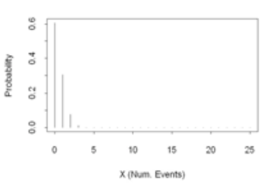

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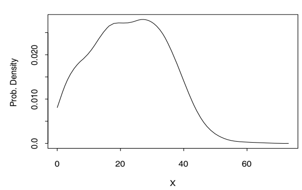

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