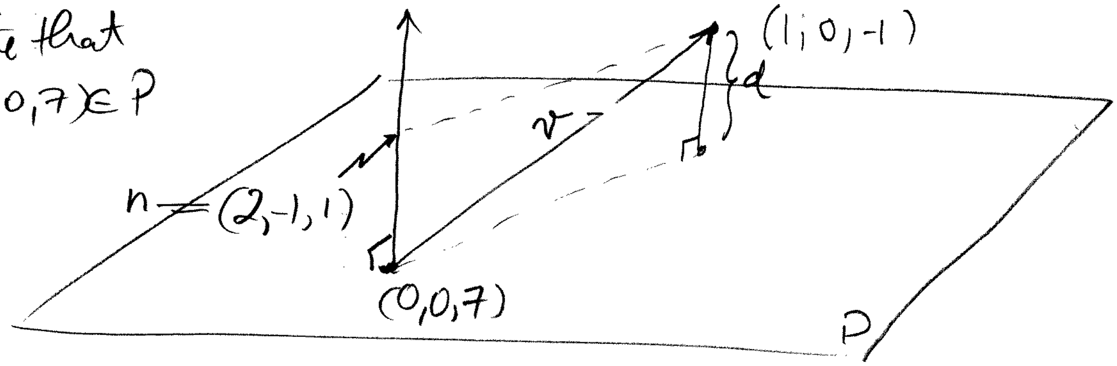
Diag F12 Solutions P

1. The distance from the point (1, 0, −1) to the plane with equation $2x - y + z = 7$ is:

(A.) $\sqrt{6}$
B. $2\sqrt{6}$
C. $\frac{\sqrt{6}}{2}$
D. $\frac{1}{6}$
E. $\frac{1}{3}$
F. $-2\sqrt{6}$

Note that $(0,0,7) \in P$



$d = \| \text{proj}_n v \| ; v = (1,0,-1) - (0,0,7) = (1,0,-8)$

$\text{proj}_n v = \dfrac{(1,0,-8)\cdot(2,-1,1)}{\|(2,-1,1)\|^2}(2,-1,1) = \dfrac{-6}{6}\cdot(2,-1,1)$

$\therefore \|\text{proj}_n v\| = \|(2,-1,1)\| = \sqrt{6}$

2. Consider the following two lines given in scalar parametric form:

$$L_1 = \{(x, y, z) \mid x = -2s + 1,\ y = s + 2, \text{ and } z = 4s + 1, \text{where } s \in \mathbf{R}\} \quad d_1 = (-2,1,4)$$

$$L_2 = \{(x, y, z) \mid x = t + 3,\ y = t + 1, \text{ and } z = t - 3, \text{where } t \in \mathbf{R}\} \quad d_2 = (1,1,1)$$

Which **one** of the following statements is correct?

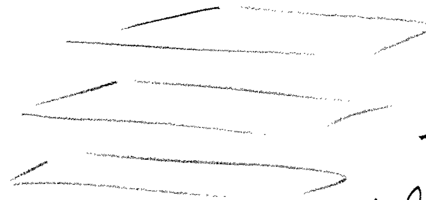
(A.) L_1 and L_2 intersect at (3, 1, −3).
B. L_1 and L_2 intersect at (−3, 1, −3).
C. L_1 and L_2 intersect at (3, −1, −3).
D. L_1 and L_2 are parallel. ×
E. L_1 and L_2 are perpendicular. ×
F. L_1 and L_2 are not coplanar. (i.e. do not intersect!)

We solve
$-2s+1 = t+3$ (1)
$s+2 = t+1$ (2)
$4s+1 = t-3$ (3)

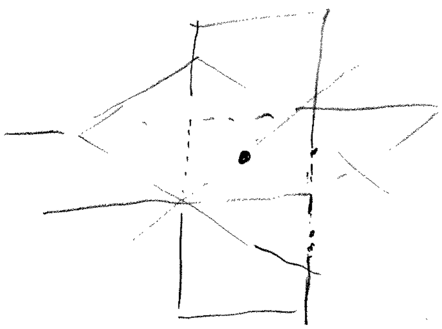
(1)−(2): $-3s-1 = 2 \therefore s=-1$, so $t=0$ (from (2)); These values of s and t satisfy (3), so L_1 & L_2 intersect at (3, 1, −3).

3. The intersection of three planes in $\mathbf{R}^3$ is <u>always</u>

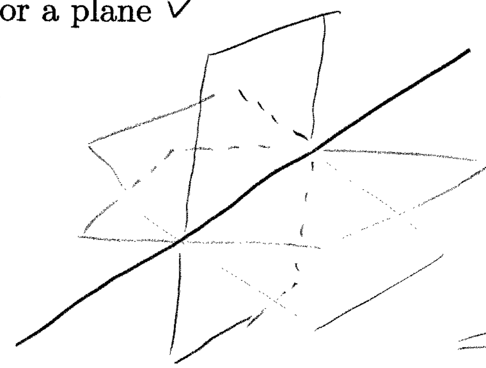
A. Empty ✗
B. A line ✗
C. A plane ✗
D. A point ✗
E. A point, or a line, or a plane ✗
F. Empty, or a point, or a line, or a plane ✓



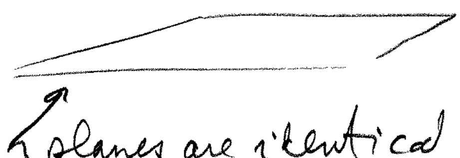
empty (planes are distinct & parallel



Point



line



planes are identical

4. The equation $5x - y + 6z = -3$ is the equation of ... a plane

A. a ~~line~~ in $\mathbf{R}^3$ with direction vector $(5, \ -1, \ 6)$.

B. a plane passing through the points $(9, \ 0, \ -8)$, $(1 \ ,1, \ 1)$ and $(0, \ 3, \ 0)$.

C. a plane ✓ with normal vector $(5, \ -1, \ 6)$ ✓ and passing through the point $(0, \ 3, \ 1)$. ✗

D. a plane with normal vector $(5, \ -1, \ 6)$ ✓ and passing through the point $(9, \ 0, \ -8)$. ✓

E. a ~~line~~ in $\mathbf{R}^3$ passing through the points $(0, \ 3, \ 0)$ and $(9, \ 0, \ -8)$

F. a plane, with normal vector $(0, \ 3, \ 0)$ and passing through the point $(5, \ -1, \ 6)$

The normal vector is $(5, -1, 6)$

$$5(9) - 0 + 6(-8) = -3 \checkmark$$

5. The volume of the parallelepiped with edges given by the vectors $u = (1, -1, 0)$, $v = (0, 1, 2)$ and $w = (2, 0, 1)$ is:

$|u \cdot v \times w|$

A. 1

B. 2

C. 3

D. 4

E. 5

F. 0

$$u \cdot v \times w = (1,-1,0) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= (1,-1,0) \cdot (1, +4, -2)$$

$$= -3$$

$\therefore$ vol $= |-3| = 3$

6. Find the area of the triangle with vertices $\overset{A}{(2, 1, 0)}$, $\overset{B}{(0, -1, 2)}$ and $\overset{C}{(1, -2, 2)}$.

A. $\sqrt{6}$

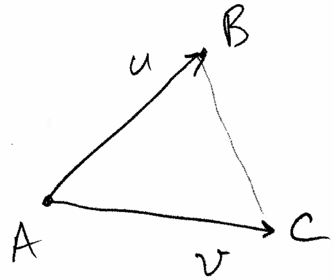
B. $2\sqrt{6}$

C. $\frac{\sqrt{6}}{2}$

D. $\frac{\sqrt{6}}{6}$

E. $\frac{\sqrt{6}}{3}$

F. $4\sqrt{6}$



area $= \frac{1}{2} \|u \times v\|$

But

$u = B - A = (-2, -2, 2)$

$v = C - A = (-1, -3, 2)$, so

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -2 & 2 \\ -1 & -3 & 2 \end{vmatrix} = (2, +2, 4). \quad \therefore \frac{1}{2}\|u \times v\| =$$

$$\frac{1}{2}\sqrt{2^2 + 2^2 + 4^2}$$

$$= \frac{1}{2}\underbrace{\sqrt{24}}_{\sqrt{4 \cdot 6}} = \frac{1}{2} \cdot 2 \cdot \sqrt{6} = \sqrt{6}$$

7. A direction vector for the line of intersection of the planes with equations $x - 2y = 1$ and $x + y - z = 0$ is:

A. (2, 1, 3)

B. (−2, 1, 3)

C. (−2, −1, 3)

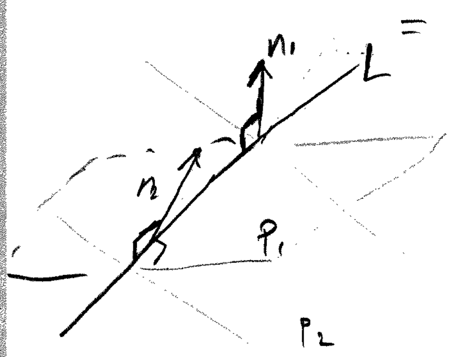
D. (2, −3, −1)

E. (−2, 3, −1)

F. (2, 3, −1)

Normals are $n_1 = (1,-2,0)$ & $n_2 = (1,1,-1)$

$\therefore$ a dir^n vector for the line of intersection is $n_1 \times$



$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (2, +1, 3)$$

($L = P_1 \cap P_2$ lies in both planes so is $\perp$ to both their normals)

8. Parametric equations of the line containing (1, −1, 2) and which is parallel to the two planes with equations $x - y = 1$ and $x + y - 3z = 0$ are:

A. $x = 1 + 3t$, $y = -1 + 3t$, $z = 2 + 2t$, $t \in \mathbf{R}$

B. $x = 1 - 3t$, $y = -1 + 3t$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

C. $x = 1 - 3t$, $y = 1 + 3t$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

D. $x = 1$, $y = -1 + 3t$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

E. $x = 1 + 3t$, $y = 1$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

F. $x = 1 - 3t$, $y = -1$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

A line parallel to these lines will have a direction vector which is perpendicular to both their normals.

Here $n_1 = (1,-1,0)$
$n_2 = (1,1,-3)$,

so $n_1 \times n_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & -3 \end{vmatrix} = (3, +3, 2)$; the only possible sol^n is A, which does pass through (1,−1,2)

9. Find a scalar equation for the plane

$$H = \{(1+s+2t, 2+t, 1+s) \mid s,t \in \mathbf{R}.\}$$

$$= \{(1,2,1) + s(1,0,1) + t(2,1,0) \mid s,t \in \mathbb{R}\}$$

A. $x - 2y - z = 4$?
B. $x + 5y - 2z = 9$ X
C. $-x + 2y + z = 4$?
D. $x - 2y + z = -3$ X
E. $-x - 2y + z = -4$ X
F. $3x + 2y - z = -2$ X

The plane will be ⊥ to both (1,0,1) and (2,1,0) and so will have normal $(1,0,1) \times (2,1,0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

$= (-1, +2, 1)$

The eqn is then $-1(x-1) + 2(y-2) + 1.(z-1) = 0$

or $-x + 2y + z = 4.$

10. If $u = (1,\ 1,\ 1)$ and $v = (2,\ 1,\ 3)$ find the orthogonal projection of u on v, that is, $\mathrm{proj}_v u$.

A. $\frac{3}{7}(2,\ 1,\ 3)$
B. $\frac{4}{7}(2,\ 1,\ 3)$
C. $\frac{3\sqrt{14}}{7}(2,\ 1,\ 3)$
D. $(1,\ 1,\ 1)$ X
E. $\frac{3}{7}(3,\ 3,\ 3)$ X
F. $2\sqrt{3}\ (1,\ 1,\ 1)$ X

$$\mathrm{proj}_v u = \frac{u \cdot v}{\|v\|^2} v$$

$$= \frac{(1,1,1)\cdot(2,1,3)}{(4+1+9)} (2,1,3)$$

$$= \frac{6}{14}(2,1,3) = \frac{3}{7}(2,1,3)$$

11. Evaluate Im(z) if

$$z = \frac{1-3i}{1+i}. \quad = \frac{(1-3i)\cdot(1-i)}{|1-i|^2}$$

$$= \frac{(1-3i)(1-i)}{1^2+(-1)^2}$$

$$= \frac{1+3-4i}{2}$$

$$= \frac{4-4i}{2} = 2-2i$$

A. 2

B. -2

C. -1

D. 1

E. 3

F. -3

12. Find the polar form of:

$$z = \frac{1+i}{1-\sqrt{3}i} = \frac{z_1}{z_2}; \quad z_1 = |z_1|e^{i\theta_1}, \quad z_2 = |z_2|e^{i\theta_2}$$

$$= \frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}$$

A. $\frac{\sqrt{2}}{2}\left(\cos(-\frac{7\pi}{12}) + i\sin(-\frac{7\pi}{12})\right)$

B. $\frac{\sqrt{2}}{2}\left(\cos(\frac{7\pi}{12}) + i\sin(\frac{7\pi}{12})\right)$

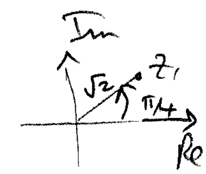
C. $\sqrt{2}\left(\cos(-\frac{\pi}{12}) + i\sin(-\frac{\pi}{12})\right)$

D. $\sqrt{2}\left(\cos(\frac{5\pi}{12}) + i\sin(\frac{5\pi}{12})\right)$

E. $\sqrt{2}\left(\cos(-\frac{5\pi}{12}) + i\sin(-\frac{5\pi}{12})\right)$

F. $\frac{\sqrt{2}}{2}\left(\cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12})\right)$

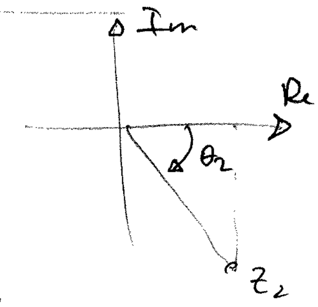
$z_1 = 1+i, \quad |z_1| = \sqrt{1^2+1^2} = \sqrt{2}$



$\cos\theta_1 = 1/\sqrt{2} = \frac{\sqrt{2}}{2}$, $\sin\theta_1 = \frac{\sqrt{2}}{2}$ $\therefore \theta_1 = \pi/4$

$z_2 = 1-\sqrt{3}i$

$|z_2| = \sqrt{1^2+(-\sqrt{3})^2} = 2$



$\cos\theta_2 = \frac{1}{2}$, $\sin\theta_2 = -\frac{\sqrt{3}}{2}$ $\therefore \theta_2 = -\pi/3$

$\therefore \theta_1 - \theta_2 = \frac{\pi}{4} + \pi/3 = \frac{3+4}{12}\cdot\pi = \frac{7\pi}{12}$ $\therefore z = \frac{\sqrt{2}}{2}e^{i\frac{7\pi}{12}}$