MAT 1330, Fall 2012 Assignment 3
Due October 12, 3:00pm at the beginning of class.
Late assignments will not be accepted; nor will unstapled assignments.

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Student Name ____________________ Student Number ____________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________

QUESTION 1. Consider the function $f(x) = \frac{1+2e^{-x}}{1-e^{-x}}$.

(a) Find $\lim_{x\to\infty} f(x)$.

Solution: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{1+2e^{-x}}{1-e^{-x}} = \lim\limits_{x\to\infty} \frac{1+2\frac{1}{e^x}}{1-\frac{1}{e^x}}$. Since $\lim\limits_{x\to\infty} 1/e^x = 0$, one gets that $\lim\limits_{x\to\infty} f(x) = \frac{1+2\times 0}{1-0} = 1$.

(b) Find $\lim_{x\to-\infty} f(x)$.

Solution: $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \frac{e^{-x}(2+e^x)}{e^{-x}(-1+e^x)} = \lim\limits_{x\to-\infty} \frac{(2+e^x)}{(-1+e^x)} = \frac{2+0}{-1+0} = -2$.

(c) Are there any infinite limits? If yes, find the left and right hand limit in each case.

Solution: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1+2e^{-x}}{1-e^{-x}} = -\infty$ because: $\lim_{x\to 0^-} 1+2e^{-x} = 3$; $\lim_{x\to 0^-} 1-e^{-x} = 0$, and $x < 0$ implies that $1-e^{-x} < 0$.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1+2e^{-x}}{1-e^{-x}} = \infty$ because: $\lim_{x\to 0^+} 1+2e^{-x} = 3$; $\lim_{x\to 0^+} 1-e^{-x} = 0$, and $x > 0$ implies that $1-e^{-x} > 0$.

QUESTION 2. Consider the function $g(x) = \begin{cases} \frac{a}{(\sin(x))^2+1} & \text{if } x < \frac{\pi}{2} \\ \frac{kx+1}{x+1} & \text{if } x \geq \frac{\pi}{2} \end{cases}$

(a) What is the condition on a and k such that g is continuous at $\pi/2$?

Solution: $\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^-} \frac{a}{(\sin(x))^2+1} = \frac{a}{2}$.

$\lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}^+} \frac{kx+1}{x+1} = \frac{1+k\frac{\pi}{2}}{\frac{\pi}{2}+1}$.

From $\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^+} f(x) = f(\pi/2)$ one gets that $\frac{a}{2} = \frac{1+k\frac{\pi}{2}}{\frac{\pi}{2}+1}$.

(b) Find a and k so that g is continuous and has the horizontal asymptote $y = 2$ as $x \to \infty$.

Solution: From $2 = \lim_{x\to\infty} f(x)$ one has that $2 = \lim_{x\to\infty} \frac{kx+1}{x+1}$, or $2 = k$. From (a) one has that $a = \frac{2+k\pi}{\frac{\pi}{2}+1} = \frac{2+2\pi}{\frac{\pi}{2}+1}$.

QUESTION 3. Using the definition of the derivative to compute $f'(x)$ where $f(x) = \frac{2}{-x-2012}$.

Solution: $\lim\limits_{h\to0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0} \frac{\frac{2}{-x-h-2012}-\frac{2}{-x-2012}}{h} =$
$\lim\limits_{h\to0} \frac{2\{-x-2012\}-2\{-x-h-2012\}}{h\{-x-h-2012\}\{-x-2012\}} =$
$\lim\limits_{h\to0} \frac{-2x-4024+2x+2h+4024}{h\{-x-h-2012\}\{-x-2012\}} =$
$\lim\limits_{h\to0} \frac{2h}{h\{-x-h-2012\}\{-x-2012\}} =$
$\lim\limits_{h\to0} \frac{2}{\{-x-h-2012\}\{-x-2012\}} = \frac{2}{(-x-2012)^2}$.

QUESTION 4. Consider the function $f(x) = |x^2 - 4|$.

(a) Find the two points at which the function is not differentiable.

Solution: The function $f(x)$ is not differentiable where $x^2 - 4 = 0$, or $(x-2)(x+2) = 0$, i.e., $x = -2, x = 2$.

(b) For the larger of the two points in (a), find the left and right limit of the slope of the function

Solution: $\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{|(2+h)^2-4|-|2^2-4|}{h} = \lim_{h\to 0^-} \frac{|4h+h^2|}{h} = \lim_{h\to 0^-} \frac{|h||4+h|}{h} = \lim_{h\to 0^-} \frac{-h|4+h|}{h} = \lim_{h\to 0^-} -|4+h| = -\lim_{h\to 0^-} 4+h = -4$

and

$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{|(2+h)^2-4|-|2^2-4|}{h} = \lim_{h\to 0^+} \frac{|4h+h^2|}{h} = \lim_{h\to 0^+} \frac{|h||4+h|}{h} = \lim_{h\to 0^+} \frac{h|4+h|}{h} = \lim_{h\to 0^+} |4+h| = \lim_{h\to 0^+} 4+h = 4$

(c) Give an explicit expression of f' wherever it is defined.

Solution: When $x \in (-2, 2)$ one has that $f(x) = 4 - x^2$, thus $f'(x) = -2x$. When $x \in (-\infty, -2) \bigcup (2, \infty)$ one has that $f(x) = -4 + x^2$, thus $f'(x) = 2x$.