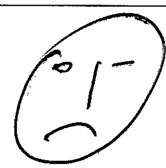
MAT 1322 S2011. TUE, MAY 31st 19:00–20:20 Prof. C. Rada

MIDTERM TEST 1

Max = 20

Sol

Student Number: ____



- Time: 80 min.

- Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted.

- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

- The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

- There are five questions worth four marks each.

1. (a) Consider the integral $\int_1^5 \frac{2}{(x-3)^{4/3}}\,dx$. Does it converge or diverge? If it converges, give its value.

$$\int_1^5 \frac{2}{(x-3)^{4/3}}\,dx = \int_1^3 \frac{2}{(x-3)^{4/3}}\,dx + \int_3^5 \frac{2}{(x-3)^{4/3}}\,dx.$$

Note that:

$$\int_1^3 \frac{2}{(x-3)^{4/3}}\,dx = \lim_{t\to 3^-} \int_1^t 2(x-3)^{-4/3}\,dx = \lim_{t\to 3} \frac{2\cdot(x-3)^{-\frac{1}{3}}}{-\frac{1}{3}}\Bigg|_1^t$$

$$= \lim_{t\to 3^-} -6\left[(t-3)^{-\frac{1}{3}} - (-2)^{-\frac{1}{3}}\right] = -\infty,$$ so Diverges.

Final answer: $\int_1^5 \frac{2}{(x-3)^{4/3}}\,dx$ is divergent

(b) Use the Comparison Test to determine if the integral $\int_1^\infty \frac{2-\cos(2011x)}{x^4+20x}\,dx$ converges or diverges.

For all $x \geq 1$ one has: $-1 \leq \cos(2011x) \leq 1$. So:

$2-\cos(2011x) \leq 2+1 = 3$. For $x \geq 1$ one has then:

$$\frac{2-\cos(2011x)}{x^4+20x} \leq \frac{3}{x^4+20x} \leq \frac{3}{x^4} \quad (\text{because } x^4 \leq x^4+20x)$$

So: Since $\int_1^\infty \frac{3}{x^4}\,dx$ is convergent ($p=4>1$), by C.T. one gets that:

$$\int_1^\infty \frac{2-\cos(2011x)}{x^4+20x}\,dx \text{ is convergent}$$

2. Sketch the region bounded by the curves $y = x^2$ and $y = 10 - x^2$. What is the area of the region?

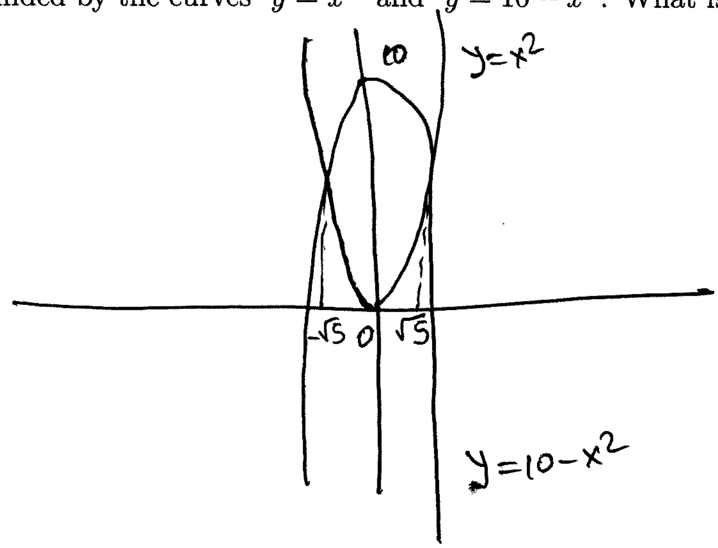
Cut points:

$x^2 = 10 - x^2$

$2x^2 = 10$

$x^2 = 5$

$x = \pm\sqrt{5}$



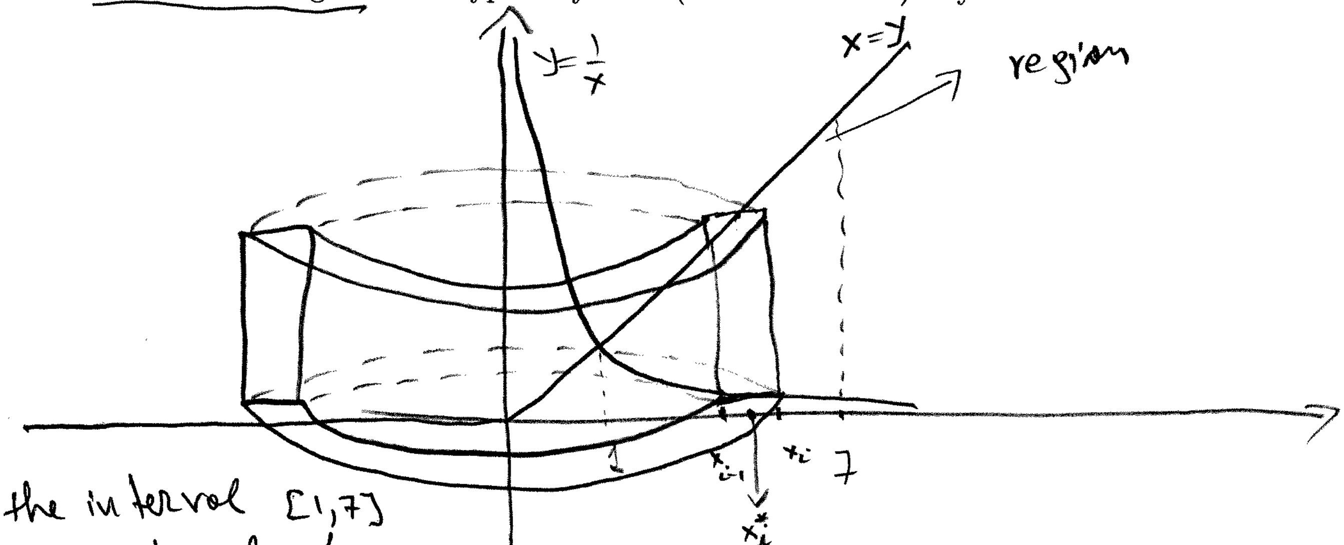
$$A = \int_{-\sqrt{5}}^{\sqrt{5}} 10 - x^2 - x^2 \, dx = \int_{-\sqrt{5}}^{\sqrt{5}} 10 - 2x^2 \, dx$$

$$= 2\int_0^{\sqrt{5}} 10 - 2x^2 \, dx \qquad \text{(by symmetry)}$$

$$= 2\left[10x - \frac{2}{3}x^3 \Big|_0^{\sqrt{5}}\right] =$$

$$2 \cdot \left[10 \cdot \sqrt{5} - \frac{2}{3}\sqrt{5} \cdot 5\right] = 2 \cdot \sqrt{5} \cdot \frac{20}{3} = \frac{40}{3}\sqrt{5}$$

3. Use the method of cylindrical shells to find the volume of the solid obtained when the region bounded by $y = x$, $y = 1/x$, $x = 1$ and $x = 7$ is rotated around the y-axis. Include a sketch of the region and a typical cylinder (with dimensions) in your solution.



Cut the interval $[1,7]$ into n subintervals of equal length: $\Delta x = \frac{7-1}{n}$. Say that the points are: $1 = x_0, x_1, \ldots, x_n = 7$. Let x_i^* be the midpoint of $[x_{i-1}, x_i]$.

$$V_i \approx \underbrace{2\pi x_i^*}_{C} \underbrace{\left(x_i^* - \frac{1}{x_i^*}\right)}_{H} \underbrace{\Delta x}_{T} \; ; \quad V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi x_i^* \left(x_i^* - \frac{1}{x_i^*}\right)\Delta x.$$

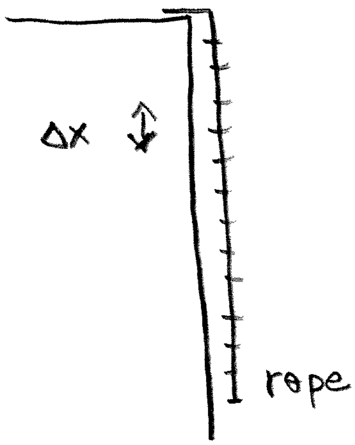
Pass to limit $(n \to \infty)$:

$$V = \int_1^7 2\pi x \left(x - \frac{1}{x}\right) dx = 2\pi \int_1^7 (x^2 - 1)\,dx =$$

$$= 2\pi \left[\frac{x^3}{3} - x \Big|_1^7 \right] = 2\pi \left[\frac{343}{3} - 7 - \left(\frac{1}{3} - 1\right) \right]$$

$$= 2\pi \left[114 - 7 + 1\right] = 2\pi (108) = \boxed{216\pi}$$

4. A heavy rope of length 18 m has a density of 1.25 kg/m and is hanging over the edge of a tall building. How much work is done pulling the rope to the top of the building? The acceleration of gravity is $g = 9.8$ m/s^2. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.



Divide the rope into n subintervals of equal length: $\Delta x = \frac{18-0}{n}$

Choose x_i^* to be a sample point in $[x_{i-1}, x_i]$.

The length of rope has a mass:

$(1.25)\Delta x$ (between x_{i-1}, x_i); so it weights: $(1.25)\Delta x \cdot g = (12.25)\Delta x$

W_i (between x_{i-1}, and x_i) is $\approx (12.25)\Delta x \cdot x_i^*$; so

$$W \approx \sum_{i=1}^{m} W_i = \sum_{i=1}^{m} (12.25)\Delta x \cdot x_i^* .$$

Pass to limit (as $n \to \infty$):

$$W = \int_0^{18} 12.25\, x\, dx = 12.25 \int_0^{18} x\, dx = 12.25\left(\frac{x^2}{2}\Big|_0^{18}\right)$$

$$= (12.25)\left(\frac{324}{2}\right) = 1984.5 \text{ (J)}$$

5. (a) Find the average value of the function $f(x) = xe^{-x}$ on the interval $[1, 2]$.

$$\frac{1}{2-1}\int_1^2 xe^{-x}dx = -xe^{-x}\Big|_1^2 - \int_1^2 -e^{-x}\cdot 1\cdot dx =$$

$f'(x) = e^{-x}$ $\quad f(x) = -e^{-x}$

$g(x) = x$ $\quad g'(x) = 1$

$$= -2e^{-2} + e^{-1} - \left[e^{-x}\Big|_1^2\right] =$$

$$= -2e^{-2} + e^{-1} - e^{-2} + e^{-1} = -3e^{-2} + 2e^{-1}$$

(b) Solve the initial value problem: $e^{-x}\dfrac{dy}{dx} = 3y\,,\ \ y(0) = e^3\,.$

$\dfrac{dy}{y} = 3e^{x}\,dx$ $\quad$ S ; I $\quad \int \frac{1}{y}\,dy = \int 3e^{x}dx$

$\ln|y| = 3e^{x} + C$; C# $\qquad |y| = e^{3e^{x}+C}$

PLUG $\quad x=0 \Rightarrow \quad e^{3} = e^{3e^{0}+C} = e^{3+C} \Rightarrow C=0$

SO $\quad |y| = e^{3e^{x}}$.

SO $\quad y(x) = e^{3e^{x}}$ $\qquad$ since $\underline{\underline{y(0)>0}}$

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• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

• There are five questions worth four marks each.

1. (a) Consider the integral $\int_1^4 \frac{3}{(x-3)^{2011/3}}\,dx$. Does it converge or diverge? If it converges, give its value.

$$\int_1^4 \frac{3}{(x-3)^{2011/3}}\,dx = \int_1^3 \frac{3}{(x-3)^{2011/3}}\,dx + \int_3^4 \frac{3}{(x-3)^{2011/3}}\,dx$$

NOTE: $\int_1^3 \frac{3}{(x-3)^{2011/3}}\,dx = \lim_{t\to 3^-} \int_1^t 3(x-3)^{-\frac{2011}{3}}\,dx = \lim_{t\to 3^-} 3 \cdot \frac{(x-3)^{-\frac{2008}{3}}}{-\frac{2008}{3}}\Big|_1^t$

$$= -\frac{9}{2008}\lim_{t\to 3^-}(t-3)^{-\frac{2008}{3}} - (-2)^{\frac{-2008}{3}} = -\infty \text{; so}$$

Diverges.

FINAL answer: $\int_1^4 \frac{3}{(x-3)^{2011/3}}\,dx$ is Divergent

(b) Use the Comparison Test to determine if the integral $\int_5^\infty \frac{3+\sin(2010x)}{x^5+23x}\,dx$ converges or diverges.

Note that for all $x \geq 5$ one has that $-1 \leq \sin(2010x) \leq 1$.

So: $3+\sin(2010x) \leq 3+1=4$. For all $x \geq 5$ one has

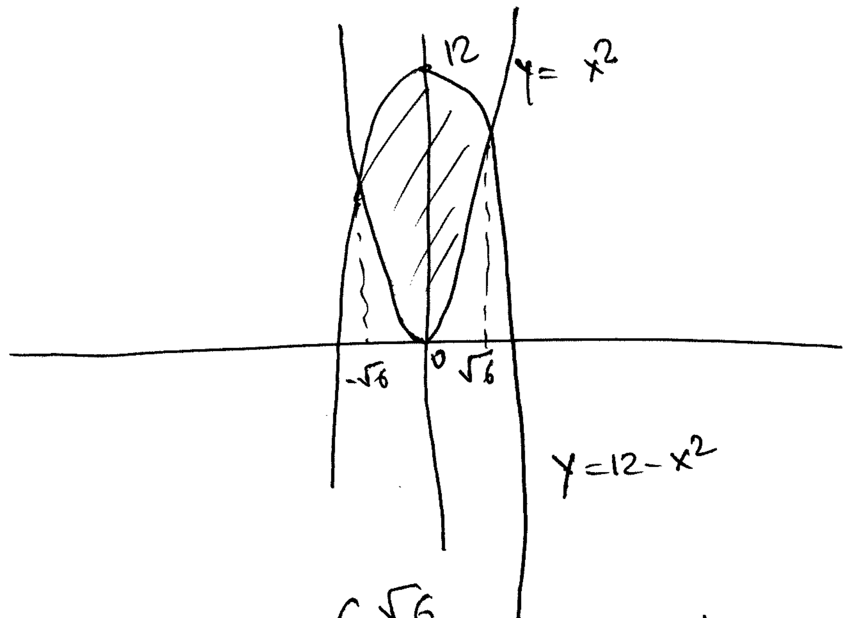
$$\frac{3+\sin(2010x)}{x^5+23x} \leq \frac{4}{x^5+23x} \leq \frac{4}{x^5} \quad (\text{since } x^5 \leq x^5+23x \text{ for } x \geq 5)$$

Since $\int_5^\infty \frac{4}{x^5}\,dx$ is convergent $(p=5>1)$,

it follows by C.T. that

$\int_5^\infty \frac{3+\sin(2010x)}{x^5+23x}\,dx$ is Convergent.

2. Sketch the region bounded by the curves $y = x^2$ and $y = 12 - x^2$. What is the area of the region?



POINTS of Intersection:

$x^2 = 12 - x^2$

$2x^2 = 12$

$x^2 = 6$

$x = \pm\sqrt{6}$

$$A = \int_{-\sqrt{6}}^{\sqrt{6}} 12 - x^2 - x^2 \, dx = \int_{-\sqrt{6}}^{\sqrt{6}} 12 - 2x^2 \, dx$$

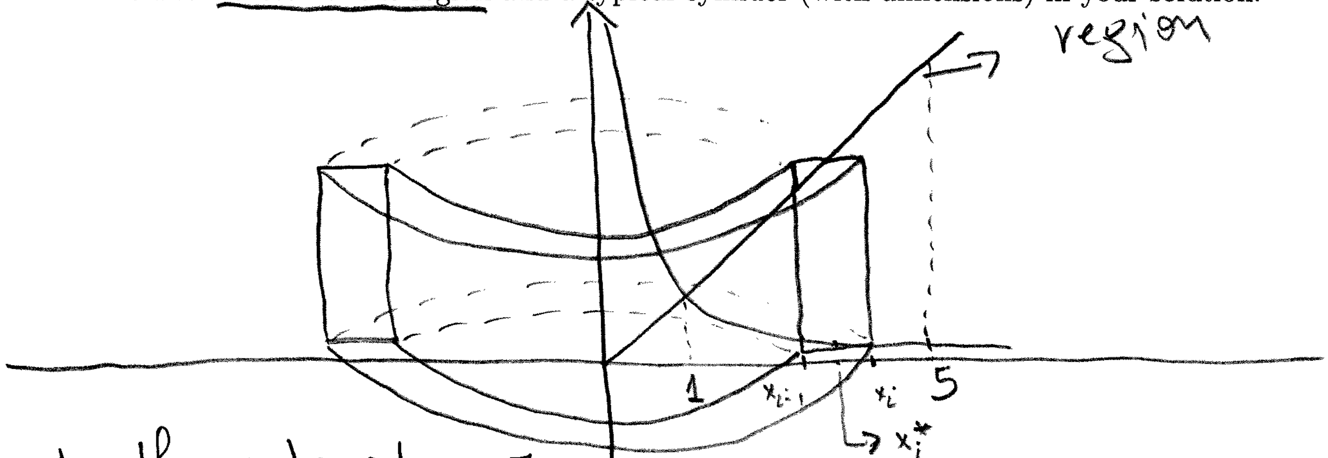
$$= 2 \int_0^{\sqrt{6}} (12 - 2x^2) \, dx$$ symmetry

$$= 2 \left[12x - 2 \cdot \frac{x^3}{3} \Big|_0^{\sqrt{6}} \right] =$$

$$= 2 \left[12 \cdot \sqrt{6} - \frac{2}{3} \cdot 6\sqrt{6} \right] = 2 \cdot 8 \cdot \sqrt{6}$$

$$= 16\sqrt{6}$$

3. Use the method of cylindrical shells to find the volume of the solid obtained when the region bounded by $y = x$, $y = 1/x$, $x = 1$ and $x = 5$ is rotated around the y-axis. Include a sketch of the region and a typical cylinder (with dimensions) in your solution.



Cut the interval $[1,5]$ into n subintervals of equal length: $\Delta x = \frac{5-1}{n}$. Say that the points of this division are: $1 = x_0, x_1, \dots, x_n = 5$. Let x_i^* be the midpoint of $[x_{i-1}, x_i]$.

$$V_i = \underbrace{2\pi x_i^*}_{C} \underbrace{\left(x_i^* - \frac{1}{x_i^*}\right)}_{H} \cdot \underbrace{\Delta x}_{T} \; ; \quad V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi x_i^* \left(x_i^* - \frac{1}{x_i^*}\right)\Delta x$$

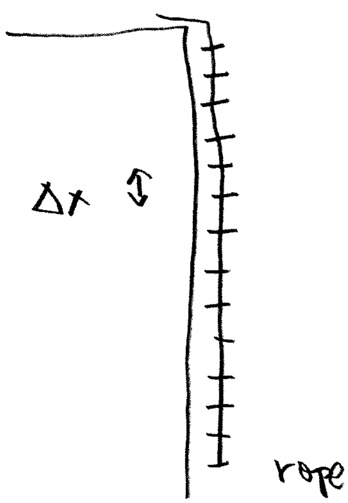
Pass to limit $(n \to \infty)$: $V = \int_1^5 2\pi x \left(x - \frac{1}{x}\right) dx =$

$$= 2\pi \int_1^5 \left(x^2 - 1\right) dx = 2\pi \left[\left. \frac{x^3}{3} - x \right|_1^5 \right] =$$

$$= 2\pi \left[\frac{125}{3} - 5 - \left(\frac{1}{3} - 1\right) \right] = 2\pi \left[\frac{125 - 1}{3} - 5 + 1 \right]$$

$$= 2\pi \left[\frac{124}{3} - 4 \right] = 2\pi \cdot \frac{112}{3} = \frac{224}{3}\pi$$

4. A heavy rope of length 16 m has a density of 1.5 kg/m and is hanging over the edge of a tall building. How much work is done pulling the rope to the top of the building? The acceleration of gravity is $g = 9.8\ \mathrm{m/s^2}$. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.



Divide the rope into n subintervals of equal length: $\Delta x = \frac{16-0}{n}$. Let the points of division be $x_0, x_1, \ldots, x_n$. Choose x_i^* to be a sample point in $[x_{i-1}, x_i]$.

The length of rope (between x_{i-1}, x_i) has mass: $(1.5)\Delta x$; so it weights: $(1.5)\Delta x \cdot g = (14.7)\Delta x$.

W_i (the work between x_{i-1} and x_i): $W_i \approx (14.7)\Delta x\, x_i^*$. So

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} (14.7)\Delta x\, x_i^*.$$

Pass to limit (as $n \to \infty$):

$$W = \int_0^{16} (14.7)\, x\, dx = 14.7 \int_0^{16} x\, dx = 14.7\left(\frac{x^2}{2}\Big|_0^{16}\right)$$

$$= 14.7\left(\frac{256}{2}\right) = 1881.6 \ \text{(J)}$$

5. (a) Find the average value of the function $f(x) = xe^x$ on the interval $[1,2]$.

$$\frac{1}{2-1}\int_1^2 xe^x\,dx = x\cdot e^x\Big|_1^2 + \left[-\int_1^2 e^x\cdot 1\,dx\right] =$$

$f' = e^x$ $\quad f(x) = e^x$

$g(x) = \cancel{x}$ $\quad g'(x) = 1$

$$= 2\cdot e^2 - 1\cdot e^1 - \left[e^x\Big|_1^2\right]$$

$$= 2\cdot e^2 - e - \left[e^2 - e^1\right]$$

$$= e^2$$

(b) Solve the initial value problem: $(e^{-x})\dfrac{dy}{dx} = 2y$, $\; y(0) = e^2$.

(S) $\dfrac{dy}{y} = e^x\cdot 2\cdot dx$ $\qquad$ (I) $\displaystyle\int \frac{1}{y}\,dy = \int 2\cdot e^x\,dx$

$\ln|y| = 2e^x + C$; $\quad C\#$ $\quad$; so $\quad |y| = e^{2e^x + C}$

PLUG $\; x=0 \;$ and get: $\quad e^2 = e^{2e^0 + C} = e^{2+C} \Rightarrow C = 0$

SO $\quad |y| = e^{2e^x}$.

Since $\quad y(0) > 0 \quad \Rightarrow \quad \boxed{y(x) = e^{2e^x}}$