MATH 3705 Final Examination
April 2003

1. $\mathcal{L}\{t^3e^{2t}\} =$

   (a) $\dfrac{6}{(s+2)^3}$

   (b) $\dfrac{6e^{-2s}}{s^3}$

   (c) $\dfrac{6}{(s-2)^3}$

   (d) $\dfrac{6}{(s-2)^4}$

   (e) None of the above.

2. $\mathcal{L}\{e^{-3t}\cos(4t)\} =$

   (a) $\dfrac{s}{(s+3)^2+16}$

   (b) $\dfrac{s+3}{(s+3)^2+16}$

   (c) $\dfrac{e^{-3s}}{s^2+16}$

   (d) $\dfrac{se^{-3s}}{s^2+16}$

   (e) None of the above.

3. $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2-2s+5}\right\} =$

   (a) $\frac{1}{2}u(t-3)e^{t-3}\sin(2t-6)$

   (b) $\frac{1}{2}u(t-3)e^{t}\sin(2t-6)$

   (c) $\frac{1}{2}u(t-3)e^{t}\sin(2t)$

   (d) $\frac{1}{2}u(t-3)e^{-3t}\sin(2t)$

   (e) None of the above.

4. $\mathcal{L}^{-1}\left\{\dfrac{3s}{(s^2+9)^2}\right\} =$

(a) $t\sin(3t)$

(b) $-t\sin(3t)$

(c) $\dfrac{1}{2}t\sin(3t)$

(d) $-\dfrac{1}{2}t\sin(3t)$

(e) None of the above.

5. The general solution of $4x^2y'' - 8xy' + 9y = 0$, valid for $x \neq 0$, is given by

(a) $c_1|x|^{3/2} + c_2|x|^{3/2}$

(b) $|x|\left[c_1\cos\left(\dfrac{\sqrt{5}}{2}\ln|x|\right) + c_2\sin\left(\dfrac{\sqrt{5}}{2}\ln|x|\right)\right]$

(c) $c_1|x| + c_2|x|^{\sqrt{5}/2}$

(d) $c_1|x|^{3/2} + c_2|x|^{3/2}\ln|x|$

(e) None of the above.

6. The general solution of $x^2y'' + xy' + (5x^2 - 9)y = 0$ near $x_0 = 0$, valid for $x > 0$, is given by

(a) $c_1J_3(\sqrt{5}x) + c_2J_{-3}(\sqrt{5}x)$

(b) $c_1J_3(\sqrt{5}x) + c_2Y_3(\sqrt{5}x)$

(c) $c_1J_{\sqrt{5}}(3x) + c_2J_{-\sqrt{5}}(3x)$

(d) $c_1J_{\sqrt{5}}(3x) + c_2Y_{\sqrt{5}}(3x)$

(e) None of the above.

7. At $x = 999$, the Fourier sine series of $f(x) = x$ on $[0, 1]$ converges to

(a) 1

(b) -1

(c) 0

(d) $\frac{1}{2}$

(e) None of the above.

8. The differential equation $4x^2y'' - 8xy' + 9\lambda y = 0$, when placed in the Sturm-Liouville form $(py')' - qy + \lambda ry = 0$, has the weight function $r(x)$ given by

(a) $\dfrac{9x^2}{4}$

(b) $-\dfrac{9x^2}{4}$

(c) $\dfrac{9x}{4}$

(d) $\dfrac{9}{4x^4}$

(e) None of the above.

9. $\mathcal{F}\{e^{3ix-|x-2|}\} =$

(a) $\dfrac{2e^{2i(\lambda+3)}}{1+(\lambda+3)^2}$

(b) $\dfrac{2e^{2i(\lambda-3)}}{1+(\lambda-3)^2}$

(c) $\dfrac{2e^{-2i(\lambda+3)}}{1+(\lambda+3)^2}$

(d) $\dfrac{2e^{-2i(\lambda-3)}}{1+(\lambda-3)^2}$

(e) None of the above.

10. $\mathcal{F}\{(x-3)e^{-(x-3)^2}\} =$

(a) $\dfrac{i\sqrt{\pi}}{2}(\lambda-3)e^{-\frac{(\lambda-3)^2}{4}}$

(b) $\dfrac{i\sqrt{\pi}}{2}(\lambda+3)e^{-\frac{(\lambda+3)^2}{4}}$

(c) $\dfrac{i\sqrt{\pi}}{2}\lambda e^{-3i\lambda-\frac{\lambda^2}{4}}$

(d) $\dfrac{i\sqrt{\pi}}{2}\lambda e^{3i\lambda-\frac{\lambda^2}{4}}$

(e) None of the above.

11. $\mathcal{F}^{-1}\{\lambda e^{-\lambda^2}\} =$

(a) xe^{-x^2}

(b) $\dfrac{1}{2\sqrt{\pi}}xe^{-\frac{x^2}{4}}$

(c) $\dfrac{i}{4\sqrt{\pi}}xe^{-4x^2}$

(d) $-\dfrac{i}{4\sqrt{\pi}}xe^{-\frac{x^2}{4}}$

(e) None of the above.

12. $\mathcal{F}^{-1}\{e^{-|\lambda|}\} =$

(a) $\dfrac{1}{\pi}\dfrac{1}{1+x^2}$

(b) $e^{-|x|}$

(c) $\dfrac{2}{1+x^2}$

(d) $\dfrac{1}{\sqrt{\pi}}e^{-x}$

(e) None of the above.

13. Solve the initial-value problem $y'' + y' - 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = -2.$

14. Consider the differential equation $xy'' - y = 0$, $x > 0$. For one (non-zero) series solution y_1 about $x_0 = 0$,

(a) find the coefficient recursion relation,

(b) solve the recursion relation.

15. Let $f(x) = \left\{ \begin{array}{ll} 1, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{array} \right\}$. Find the full Fourier series of f on $[0, 2]$.

16. The solution of the wave equation $u_{xx} = \dfrac{1}{c^2}u_{tt}$, $0 < x < L$, satisfying the boundary conditions $u(0,t) = u(L,t) = 0$, has the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right].$$

Find the solution of $u_{xx} = u_{tt}$, $0 < x < \pi$, which also satisfies the initial conditions $u(x,0) = 0$, $u_t(x,0) = x(\pi - x)$.

17. The bounded solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = a$ has the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

Find the bounded solution of Laplace's equation outside the circle $r = 1$, subject to the boundary condition $u(1,\theta) = 1 - \cos(2\theta) + \sin\theta$.

18. Find all eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem

$y'' + \lambda y = 0,\ y'(0) = 0,\ y(1) = 0.$

Table of Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,\ s > 0$$

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}},\ p > -1,\ \text{and } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),\ s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s),\ s > a \geq 0$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0),\ n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n}F(s),\ n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)\,dx$$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}\{f(t) * g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x)\,dx\right\} = F(s)G(s),\ \text{where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as},\ a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-\omega s}}\int_0^\omega e^{-st}f(t)\,dt \text{ whenever } f \text{ is periodic with period } \omega$$

Summary of Fourier Series

1. The Fourier sine series of a function f defined on $[0, L]$ is given by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

2. The Fourier cosine series of a function f defined on $[0, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0.$$

3. The full Fourier series of a function f defined on $[-L, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

4. The full Fourier series of a function f defined on $[0, 2L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0, \qquad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

5. The Fourier series of an ω-periodic function f on $\mathbb{R} = (-\infty, \infty)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{\omega}\right) + b_n \sin\left(\frac{2n\pi x}{\omega}\right)\right],$$

$$\begin{aligned}
a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\omega}\int_0^{\omega} f(x)\cos\left(\frac{2n\pi x}{\omega}\right)dx \\
&= \frac{2}{\omega}\int_{\alpha}^{\alpha+\omega} f(x)\cos\left(\frac{2n\pi x}{\omega}\right)dx, \quad n \geq 0, \\
b_n &= \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\omega}\int_0^{\omega} f(x)\sin\left(\frac{2n\pi x}{\omega}\right)dx \\
&= \frac{2}{\omega}\int_{\alpha}^{\alpha+\omega} f(x)\sin\left(\frac{2n\pi x}{\omega}\right)dx, \quad n \geq 1,
\end{aligned}$$

where $L = \frac{\omega}{2}$ and α is any real number.

Table of Fourier Transforms

$$\mathcal{F}\{f(x)\} = \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{i\lambda x} dx$$

$$\mathcal{F}^{-1}\{F(\lambda)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x}\, d\lambda$$

$$\mathcal{F}\left\{u(x-a) - u(x-b)\right\} = \frac{1}{i\lambda}\left[e^{i\lambda b} - e^{i\lambda a}\right],\ a < b$$

$$\mathcal{F}\{e^{-|x|}\} = \frac{2}{1+\lambda^2}$$

$$\mathcal{F}\{\delta(x-a)\} = e^{i\lambda a}$$

$$\mathcal{F}\{e^{iax} f(x)\} = \widehat{f}(\lambda + a)$$

$$\mathcal{F}\{f(x-a)\} = e^{i\lambda a}\widehat{f}(\lambda)$$

$$\mathcal{F}\{f'(x)\} = -i\lambda \widehat{f}(\lambda)$$

$$\mathcal{F}\{x f(x)\} = -i\frac{d\widehat{f}}{d\lambda}$$

$$\mathcal{F}\left\{f(\alpha x)\right\} = \frac{1}{|\alpha|}\widehat{f}\left(\frac{\lambda}{\alpha}\right),\ \alpha \neq 0$$

$$\mathcal{F}\{e^{-tx^2}\} = \sqrt{\frac{\pi}{t}}\, e^{-\frac{\lambda^2}{4t}},\ t > 0$$

$$\mathcal{F}\{(f * g)(x)\} \equiv \mathcal{F}\left\{\int_{-\infty}^{\infty} f(s)g(x-s)ds\right\} = \widehat{f}(\lambda)\widehat{g}(\lambda),\ \text{where } \widehat{g}(\lambda) = \mathcal{F}\{g(x)\}$$