MAT 1341E Assignment 1, 2011

Due: 11:30 am, 6-October, 2011.
Instructor: Sanghoon Baek

Family Name:______________________

First Name:________________________

Student number:____________________

(For the marker's use only →)

1	
2	
3	
4	
5	
6	
7	
[Bonus] 8	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 6 to 8, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
2. Questions 1 to 5 are worth 1 point each, and no part marks will be given. However, you must show some work to obtain the point. Simply writing the correct answer will earn you 0.
3. Questions 6 and 7 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.** Question 8 is a bonus question and is worth 4 points. Earning points here will be much more difficult than in questions 1-7.
4. Homework is collected at the beginning of the lecture. **No late assignments will be accepted**.

1. (1 mark) Which two of the following subsets of $\mathbb{R}^4$ are closed under the usual operation of multiplication by scalars?

A. $\{(a,\ b,\ c,\ d) |\ ab = 0\}$
B. $\{(a,\ b,\ c,\ d) |\ a = 1, b = 0 \text{ and } c + d = 1\}$
C. $\{(a,\ b,\ c,\ d) |\ a > 0 \text{ and } b < 0\}$
D. $\{(a,\ b,\ c,\ d) |\ a > 0 \text{ and } b > 0\}$
E. $\{(a,\ b,\ c,\ d) |\ a + b + c + d = 0\}$

A. If $ab = 0$, then $kab = 0$ for any $k \in \mathbb{R}$. Hence, this set is closed under the scalar multiplication.
B. $2(1, 0, 1, 0) = (2, 0, 2, 0) \notin \{(a,\ b,\ c,\ d) |\ a = 1, b = 0 \text{ and } c + d = 1\}$.
C. $-1(1, -1, 0, 0) = (-1, 1, 0, 0) \notin \{(a,\ b,\ c,\ d) |\ a > 0 \text{ and } b < 0\}$.
D. $-1(1, 1, 0, 0) = (-1, -1, 0, 0) \notin \{(a,\ b,\ c,\ d) |\ a > 0 \text{ and } b > 0\}$.
E. If $a + b + c + d = 0$, then $k(a + b + c + d) = 0$ for any $k \in \mathbb{R}$. Hence, this set is closed under the scalar multiplication.

ANSWER A, E

2. (1 mark) Which two of the following are subspaces of $F[0, 1] = \{f \mid f : [0, 1] \to \mathbb{R}\}$?

$U = \{f \in F[0, 1] \mid f(0)f(1) = 0\}$
$V = \{f \in F[0, 1] \mid f(0) + f(1) = 0\}$
$S = \{f \in F[0, 1] \mid f(x) = -2f(x), \forall x \in [0, 1]\}$
$T = \{f \in F[0, 1] \mid f(1) \leq 0\}$

U: Let $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$ and $g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$ for $0 \leq x \leq 1$. Then $f, g \in U$. But $(f + g)(0)(f + g)(1) = (f(0) + g(0))(f(1) + g(1)) = 1 \neq 0$. Hence, $f + g \notin U$, i.e., U is not a subspace.

V: If $f, g \in V$, then $(f + g)(0) + (f + g)(1) = f(0) + g(0) + f(1) + g(1) = 0$. For any $k \in \mathbb{R}$ and any $f \in V$, $kf(0) + kf(1) = k(f(0) + f(1)) = 0$. Hence, V is a subspace.

S: If $f, g \in S$, then $(f + g)(x) = f(x) + g(x) = -2f(x) - 2g(x) = -2(f + g)(x)$. For any $k \in \mathbb{R}$ and any $f \in S$, $kf(x) = -2kf(x)$. Therefore, S is a subspace.

T: Let $f(x) = -1$ for all $x \in [0, 1]$. Then $f \in T$. But $-f(x) \notin T$. Therefore, this set is not a subspace.

ANSWER V, S

3. (1 mark) Which <u>three</u> of the following statements are true?

I. The span of two distinct vectors u and v in $\mathbb{R}^3$ is a plane through the origin.

II. The span of a single nonzero vector u in $\mathbb{R}^2$ is a line.

III. A set of vectors $\{u, v, w\} \subseteq X$ spans a vector space X if every $x \in X$ is a linear combination of v and w.

IV. Any spanning set for $\mathbb{R}^2$ contains at least two elements.

V. $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\}$ spans $M_{2,2}(\mathbb{R})$.

I. Let $u = 0$ and v be any nonzero vector. Then the span of these two vectors is v, which is a line.

II. $\mathbb{R}u$ is a line.

III. If $X = \text{span}\{v, w\}$ then $X = \text{span}\{u, v, w\}$.

IV. Assume that $\mathbb{R}^2 = \mathbb{R}v$ for some nonzero vector v. Then $2 = \dim(\mathbb{R}^2) = \dim(\mathbb{R}v) = 1$. This is a contradiction. Therefore, any spanning set of $\mathbb{R}^2$ should contain at least two elements.

V. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\}$

ANSWER: II,III,IV

4. (1 mark) Which <u>three</u> of the following sets are linearly independent?

A. $\{(1,\ -5,\ 2)\}$

B. $\{(5,\ 1,\ 0),\ (0,\ 0,\ 0),\ (-2,\ 0,\ 1)\}$

C. $\{(5,\ 1,\ 0),\ (2,\ 2,\ 2)\}$

D. $\{(0,\ 0,\ 0)\}$

E. $\{(5,\ 1,\ 0),\ (-2,\ 0,\ 1),\ (3,\ 2,\ 1)\}$

As any set containing the zero vector is linearly dependent, the only candidates are A, C and E. A is linearly independent as it contains only a nonzero vector. If $a(5,\ 1,\ 0) + b(2,\ 2,\ 2) = 0$, then $a = b = 0$. Hence C is linearly independent. If $a(5,\ 1,\ 0) + b(-2,\ 0,\ 1) + c(3,\ 2,\ 1) = 0$, then $a = b = c = 0$. Hence, E is linearly independent.

ANSWER: A, C, E

5. (1 mark) If we give $W = \mathbb{R}^2$ the *non-standard* operations

$$(x,\ y) \oplus (x',\ y') = (x + x' + 9,\ y + y' - 5) \quad (\text{ vector addition})$$

and

$$k \otimes (x,\ y) = (kx + 9k - 9,\ ky - 5k + 5) \quad (\text{multiplication by scalars}),$$

then W *is* a vector space over $\mathbb{R}$.

- (0.5 mark) What is the zero vector of W?
- (0.5 mark) If $\mathbf{v} = (x,\ y)$ is in W then what is $-\mathbf{v}$?

$$0 = 0 \otimes (x,\ y) = (0 + 0 - 9,\ 0 + 0 + 5) = (-9, 5).$$

$$-v = -1 \otimes (x,\ y) = (-x - 9 - 9, -y + 5 + 5) = (-x - 18, -y + 10).$$

ANSWER $0 = (-9, 5)$

$-\mathbf{v} = (-x - 18, -y + 10)$

6. (6 marks) Let $u = (3, -2, 1)$, and $U = \{w \in \mathbb{R}^3 \mid w \cdot u = 0\}$

(a) (2 marks) Is U a subspace of $\mathbb{R}^3$?

(b) (2 marks) Find a spanning set for U.

(c) (2 marks) Give a complete geometric description of U.

(*You must justify your answers.*)

(a) If $v, w \in U$, then $(v + w) \cdot u = v \cdot u + w \cdot u = 0 + 0 = 0$. For any $k \in \mathbb{R}$ and any $w \in U$, $(kw) \cdot u = k(w \cdot u) = k0 = 0$. Hence, U is a subspace of $\mathbb{R}^3$.

(b) Let $(x, y, z) \in U$. Then $3x - 2y + z = 0$. Therefore, $(x, y, z) = (x, y, 2y - 3x) = x(1, 0, -3) + y(0, 1, 2)$ and thus $\{(1, 0, -3), (0, 1, 2)\}$ is a spanning set.

(c) U is the plane through the origin with normal vector u.

Space for problem 6

7. (6 marks) Consider the vector space $F(\mathbb{R}) = \{f \mid f\colon\mathbb{R} \to \mathbb{R}\}$, with the standard operations. Recall that the zero of $F(\mathbb{R})$ is the function that has the value 0 for all $x \in \mathbb{R}$.

Let $W = \{f \in F(\mathbb{R}) \mid f(0) = f(\pi)\}$ be the subspace of functions which have the same value at $x = 0$ and $x = \pi$.

(a) (2 marks) Show that $\cos 2x$ belongs to W and show that $\cos x$ does not belong to W.
(b) (1 mark) Show that $\sin^2 x \in \operatorname{span}\{1, \cos 2x\}$.
(c) (1 mark) Show that $\sin^2 x \notin \operatorname{span}\{1, \sin 2x\}$.
(d) (2 marks) By any means, show that $\{1, \sin^2 x, \sin 2x\}$ is linearly independent.

(*You must justify your answers.*)

(a) As $\cos 0 = 1$ and $\cos 2\pi = 1$, $\cos 2x \in W$. As $\cos\pi = -1$, $\cos x \notin W$.
(b) As $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$, we have $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.
(c) Suppose that $\sin^2 x = a + b\sin 2x$ for some $a, b \in \mathbb{R}$. If $x = \pi$, then $a = 0$. If $x = \frac{\pi}{2}$, then $0 = b\sin\pi = \sin^2\frac{\pi}{2} = 1$, which is a contradiction. Hence, $\sin^2 x \notin \operatorname{span}\{1, \sin 2x\}$.
(d) Suppose that $a + b\sin^2 x + c\sin 2x = 0$ for some $a, b, c \in \mathbb{R}$. If $x = \pi$, then $a = 0$. If $x = \frac{\pi}{2}$, then $b = 0$. If $x = \frac{\pi}{4}$, then $c = 0$. Hence, $\{1, \sin^2 x, \sin 2x\}$ is linearly independent.

Space for problem 7

8. [Bonus](4 marks) Let $E = \{\text{“}ax + by + cz = d\text{”} \mid a, b, c, d \in \mathbb{R}\}$ be the set of linear equations with real coefficients in the variables x, y and z. Equip E with the usual operations on equations that you learned in high school: addition of equations, denoted here by “$\oplus$” and multiplication by scalars, denoted here by “$\circledast$”, as follows:

$$\text{“}ax + by + cz = d\text{”} \oplus \text{“}ex + fy + gz = h\text{”} := \text{“}(a + e)x + (b + f)y + (c + g)z = d + h\text{”}$$

and

$$\forall k \in \mathbb{R}, \quad k \circledast \text{“}ax + by + cz = d\text{”} := \text{“}ka\,x + kb\,y + kc\,z = k\,d\text{”}.$$

You may assume without proof that E is a vector space.

(a) (2 marks) Find the zero vector of E.
(b) (2 marks) Find a spanning set for E.

(*You must justify your answers.*)

(a) $0 \circledast \text{“}ax + by + cz = d\text{”} := \text{“}0 + 0 + 0 = 0\text{”}$. Hence, “$0 = 0$” is the zero vector.

(b) Note that $a \circledast \text{“}x = 0\text{”} := \text{“}ax = 0\text{”}$. Similarly, $b \circledast \text{“}y = 0\text{”} := \text{“}by = 0\text{”}$, $c \circledast \text{“}z = 0\text{”} := \text{“}cz = 0\text{”}$, and $d \circledast \text{“}0 = 1\text{”} := \text{“}0 = d\text{”}$. Hence,

$$\text{“}ax + by + cz = d\text{”} = (a \circledast (\text{“}x = 0\text{”})) \oplus (b \circledast (\text{“}y = 0\text{”})) \oplus (c \circledast (\text{“}z = 0\text{”})) \oplus (d \circledast (\text{“}0 = 1\text{”})).$$

Therefore, $\{\text{“}x = 0\text{”}, \text{“}y = 0\text{”}, \text{“}z = 0\text{”}, \text{“}0 = 1\text{”}\}$ is a spanning set.

Space for problem 8