Calculus II

LECTURE NOTES

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(These Lecture Notes replace neither the Text Book nor the Lectures)

INDETERMINATE FORMS and L'HOSPITAL'S RULE.

Indeterninate Forms and L'Hospital's Rule

There are seven indeterminate forms:

Indeterminate forms	Example
∞/∞	$\lim\limits_{x\longrightarrow\infty} \frac{\ln x}{x^2}$
$0/0$	$\lim\limits_{x\longrightarrow 0} \frac{\sin x - x}{x^3}$
$\infty - \infty$	$\lim\limits_{x\longrightarrow\infty} (xe^{1/x} - x)$
$\infty \cdot 0$	$\lim\limits_{x\longrightarrow -\infty} (xe^x)$
0^0	$\lim\limits_{x\longrightarrow 0^+} (\sin x)^{\tan x}$
∞^0	$\lim\limits_{x\longrightarrow\infty} (e^x + x)^{1/x}$
1^∞	$\lim\limits_{x\longrightarrow\infty} \left(1 + \frac{1}{x}\right)^x$

For the first two types, we can use the L'Hospital's Rule:

L'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty.$$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\pm\infty$).

Example 1: $\lim_{x \to 0} \frac{\sin x}{x} = ?$

Example 2: $\lim_{x \to 0} \frac{1 - \cos x}{x} = ?$

Example 3: $\lim_{x \to \infty} \frac{e^x}{x^2} = ?$

Example 4: $\lim_{x \to 0} \frac{\sin x - x}{x^3} = ?$

Example 5: $\lim_{x\to 0} \frac{\tan x - x}{x^3} = ?$

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Example 6: $\lim_{x\to\infty} \frac{x - \cos x}{x} = ?$

Example 7: $\lim_{x\to-\infty}\dfrac{\sqrt{2x^2+3x+2}}{1-x}$ =?

Solution: We have indeterminate form of $\dfrac{\infty}{\infty}$.
By L'Hospital's rule we have

$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+3x+2}}{1-x}=\lim_{x\to-\infty}\frac{-(4x+3)}{2\sqrt{2x^2+3x+2}},$$

which is still of the form $\dfrac{\infty}{\infty}$.
Further applications of the L'Hospital's rule do not make this limit simple.
It is better to evaluate this limit as follows:

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{2x^2+3x+2}}{1-x} &= \lim_{x\to-\infty}\frac{\sqrt{x^2(2+\frac{3}{x}+\frac{2}{x^2})}}{x(\frac{1}{x}-1)}\\ &= \lim_{x\to-\infty}\frac{|x|\sqrt{2+\frac{3}{x}+\frac{2}{x^2}}}{x(\frac{1}{x}-1)}\\ &= \lim_{x\to-\infty}\frac{-x\sqrt{2+\frac{3}{x}+\frac{2}{x^2}}}{x(\frac{1}{x}-1)}\\ &= \lim_{x\to-\infty}\frac{-\sqrt{2+\frac{3}{x}+\frac{2}{x^2}}}{\frac{1}{x}-1}=\frac{-\sqrt{2}}{-1}=\sqrt{2}.\end{aligned}$$

Homework: Verify the following limits.

1. $\lim_{x\to0}\dfrac{2\sin x-\sin(2x)}{2e^x-2-2x-x^2}=3$

2. $\lim_{x\to1}\dfrac{\ln x}{x^2-1}=1/2$

3. $\lim_{x\to0}\dfrac{\tan 2x-2x}{x-\sin x}=16.$

4. $\lim_{x\to0}\dfrac{x^2\sin(\frac{1}{x})}{x}=0.$

If we have an indeterminate form of

$$\infty - \infty \quad \text{or} \quad 0 \cdot \infty,$$

we put it in the form of

$$\infty/\infty \quad \text{or } 0/0,$$

and then we apply the L'Hospital's rule.

$$\left.\begin{aligned} &\lim_{x\to(\pi/2)^-} (\sec x - \tan x) = 0 \\ &\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = 1/2 \\ &\lim_{x\to 0}\left(\frac{1}{x^4} - \frac{1}{x^2}\right) = \infty \end{aligned}\right\} \qquad (\infty - \infty)$$

$$\left.\begin{aligned} &\lim_{x\to\infty} x^3 \cdot e^{-x^2} = 0 \\ &\lim_{x\to\infty}\left(\frac{1}{x}\right) e^x = \infty \\ &\lim_{x\to\pi}(x-\pi)\cot x = 1 \end{aligned}\right\} \qquad (\infty \cdot 0)$$

Example 1: $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) =?$

Example 2: $\lim_{x \to \infty} x^3 \cdot e^{-x^2} = ?$

Example 3: $\lim_{x \to 0^+} x^2 \ln x = ?$

Indeterminate Powers: $0^0, \infty^0, 1^\infty$

Each of these three cases can be treated by taking natural logarithm:
Suppose

$$\lim_{x\to a}[f(x)]^{g(x)}$$

is one of the form of $0^0,\ \infty^0,\ 1^\infty$.
Let $y=[f(x)]^{g(x)}$. Then $\ln y = g(x)\ln f(x)$
will have the indeterminate product of the form $0\cdot\infty$.

$y=\lim_{x\to a}[f(x)]^{g(x)}$	$\ln y = g(x)\ln f(x)$
0^0	$0\cdot(-\infty)$
∞^0	$0\cdot\infty$
1^∞	$\infty\cdot 0$

$$\left.\begin{aligned}&\lim_{x\to 0^+} x^x = 1\\ &\lim_{x\to 0^+} x^{\sin x} = 1\\ &\lim_{x\to 0^+} (\sin x)^{3/\ln x} = e^3\\ &\lim_{x\to 0^+} x^{\ln a/(1+\ln x)} = a, \quad (a>0)\end{aligned}\right\}\ (0^0)$$

$$\left.\begin{aligned}&\lim_{x\to\infty} x^{1/x} = 1\\ &\lim_{x\to\infty} (e^x+x)^{1/x} = e\\ &\lim_{x\to\infty} x^{\ln a/(1+\ln x)} = a, \quad (a>0)\end{aligned}\right\}\ (\infty^0)$$

$$\left.\begin{aligned}&\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\\ &\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\sqrt{x}} = 0\\ &\lim_{x\to 0}(\cos\ 2x)^{1/x^2} = e^{-2}\\ &\lim_{x\to 0}(\cos\ 3x)^{5/x} = 1\\ &\lim_{x\to 0}(x+1)^{\frac{\ln a}{x}} = a, \quad (a>0)\end{aligned}\right\}\ (1^\infty)$$

Remark: One should recognize the following forms as "determinate"!

$$\begin{aligned}
&\infty + \infty \longrightarrow \infty \\
&-\infty - \infty \longrightarrow -\infty \\
&0^{\infty} \longrightarrow 0 \\
&0^{-\infty} \longrightarrow \infty
\end{aligned}$$

Example 1: $\lim_{x \to 0^+} (\sin x)^{\tan x} =?\ (0^0)$

Solution: Set $y = (\sin x)^{\tan x} \Longrightarrow \ln y = \ln\left((\sin x)^{\tan x}\right) = \tan x \ln(\sin x)$.

$$\begin{aligned}
\lim_{x \to 0^+} \ln y &= \lim_{x \to 0^+} \tan x \ln(\sin x) \\
&= \lim_{x \to 0^+} \frac{\ln(\sin x)}{1/\tan x} \\
&= \lim_{x \to 0^+} \frac{\ln(\sin x)}{\cot x} \qquad \left(\frac{-\infty}{\infty}\right) \\
&\overset{L'H}{=} \lim_{x \to 0^+} \left(\frac{\frac{\cos x}{\sin x}}{-\csc^2 x}\right) \\
&= \lim_{x \to 0^+} \frac{\cos x}{\sin x} \cdot (-\sin^2 x) \\
&= \lim_{x \to 0^+} (-\cos x \cdot \sin x) = 0.
\end{aligned}$$

$$\lim_{x \to 0^+} (\sin x)^{\tan x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^0 = 1.$$

Example 2: $\lim_{x\to 0}(\cos\ 2x)^{1/x^2} =?\ (1^\infty)$

Solution:
$y = (\cos\ 2x)^{1/x^2} \Longleftrightarrow \ln y = \left(\frac{1}{x^2}\right) \cdot \ln(\cos 2x)$.

$$\begin{aligned} \lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{\ln(\cos 2x)}{x^2} \\ &\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{-2\sin 2x}{\cos 2x}}{2x} \\ &= (-2) \lim_{x\to 0} \left(\frac{\sin 2x}{2x} \cdot \frac{1}{\cos 2x}\right) = -2 \end{aligned}$$

$$\boxed{\lim_{x\to 0}(\cos\ 2x)^{1/x^2} = e^{-2}.}$$

Example 3: $\lim_{x\to\infty}(e^x + x)^{1/x} =?\ (\infty^0)$

Solution: $y = (e^x + x)^{1/x} \Longleftrightarrow \ln y = \frac{1}{x} \cdot \ln(e^x + x)$.

$$\begin{aligned} \lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \\ &\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x+x}(e^x + 1)}{1} \\ &= \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \\ &\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{e^x + 1} \\ &\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1. \end{aligned}$$

$$\boxed{\lim_{x\to\infty}(e^x + x)^{1/x} = e^1 = e.}$$

Homework: Show that $\lim_{x\to 0}(x + 1)^{\frac{\ln a}{x}} = a$, if $a > 0$.