Physics 1003/BIT 1203
Fall 2012

Lecture 17

Special Relativity

Rotational Motion

Table 10-1

Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Equation Number	Linear Equation	Missing Variable		Angular Equation	Equation Number
(2-11)	$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$	(10-12)
(2-15)	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$	(10-13)
(2-16)	$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10-14)
(2-17)	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$	(10-15)
(2-18)	$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$	(10-16)

- Replacing the linear variables with their rotational equivalents in the kinematic equations produces the rotational kinematic equations

$x - x_0$	$\theta - \theta_0$
v	ω
t	t
a	α
v_0	ω_0

Links between Linear Motion and Angular Motions

- The arc length s, traced out by a rotation θ is

$$s = r\theta$$ Equation 10-17

- The linear speed v (tangential to the rotation)

$$v = r\omega$$ Equation 10-18

- The linear acceleration (tangential to the rotation)

$$a_t = r\alpha$$ Equation 10-22

Period and Frequency

- As already discussed (in uniform circular motion) for rotations with a constant angular velocity:

Period $$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$ Eqn. 10-19

Frequency $$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Relationship between Angular and Linear Variables

•All of the variables for tangential linear motion of a particle in the rigid body are related to the angular variables

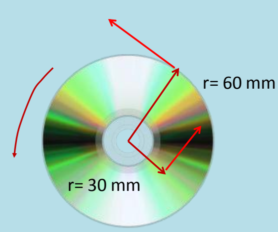
$$s = r\theta$$

$$v = r\omega$$

$$a = r\alpha$$

Linear Motion

Rotational Motion



The rotation is counter clockwise, so the angular velocity is positive

- Suppose this CD is rotating at 2.0 rev/s
- Calculate the tangential linear velocity of a particle at distances (a) r = 30 mm and (b) 60 mm from the centre of rotation

- First convert rev/s to radians/sec

$$\omega = +2.0\ \text{rev/s} = +2.0\ \text{rev/s} \times \left(\frac{2\pi\ \text{radians}}{1\ \text{rev}}\right) = +4\pi\ \text{rad/s}$$

It's okay to leave pi in an intermediate result, but for the final answer to a calculated problem, it should be multiplied out

- Now use equation 10-17 to solve:

$$v = r\omega$$

The units of v will depend on the units used to measure r

- For r = 30 mm

$$v = r\omega = (0.03\ \text{m})(+4\pi\ \text{rad/s}) = +0.38\ \text{m/s}$$
$$\text{v} = +0.4\ \text{m/s to 1 s.f.}$$

- For r = 60 mm

$$v = r\omega = (0.06\ \text{m})(+4\pi\ \text{rad/s}) = +0.75\ \text{m/s}$$

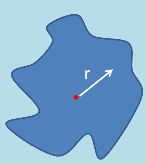
$$v = +0.8\ \text{m/s to 1 s.f.}$$

- The point twice as far from the centre of rotation has twice the tangential velocity

Rotational Kinetic Energy

- Imagine that we have a rotating rigid body made up of a large number of particles.
- Each particle has a linear kinetic energy K_i

$$K = \sum K_i$$



Each particle has a mass m_i and a linear velocity v_i

$$K = \sum \frac{1}{2} m_i v_i^2$$

Total KE is the sum of individual KEs

- Each particle is a distance r_i from the centre of rotation, so we can write

$$v_i = r_i \omega_i$$

$$K = \sum \frac{1}{2} m_i v_i^2$$

$$K = \sum \frac{1}{2} m_i (r_i \omega_i)^2$$

- As long as the body is rigid, ω must be the same for all particles

$$K = \sum \frac{1}{2} m_i (r_i \omega_i)^2 \Rightarrow K = \frac{1}{2}\omega^2 \sum m_i r_i^2$$

- We define the Rotational Inertia (or Moment of Inertia), I as

$$I = \sum m_i r_i^2 \qquad \text{Eqn. 10.33}$$

- The rotational kinetic energy is now

$$K = \frac{1}{2} I \omega^2 \qquad \text{Eqn. 10.34}$$

Rotational Inertia

- The rotational inertia for a rotating body is given by the expression

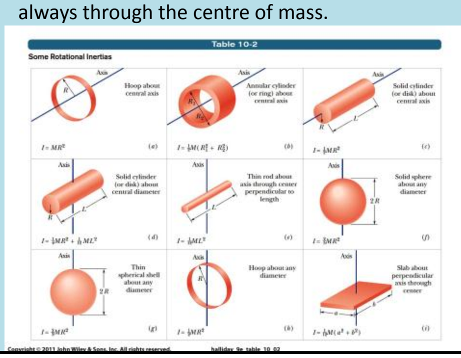
$$I = \sum m_i r_i^2 \qquad \text{Eqn. 10-34}$$

- Which for a continuous mass distribution can be written as

$$I = \int r^2 dm \qquad \text{Eqn. 10-35}$$

- It is the equivalent quantity in rotational moment to the mass for linear motion

- For well defined geometric shapes, the integrals can be evaluated to give general expressions. The axis is always through the centre of mass.



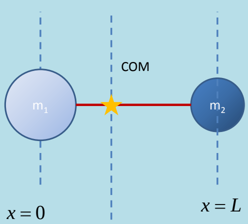
Rotational Inertia Calculation for an Object Composed of Discrete Particles

- This is where we use the summation definition

$$I = \sum m_i r_i^2 \qquad \text{Eqn. 10-34}$$

This Dumbbell has two point masses separated by a massless rod

Rotational Inertia of a Dumbbell



$$x_{COM} = \frac{m_2}{m_1 + m_2} L$$

Treat the masses as point masses

The rotation axis goes through the centre of mass

First, calculate the centre of mass!

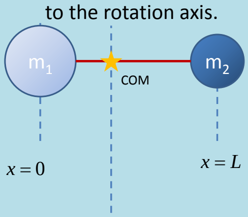
- First find the centre of mass
- Set the origin x= 0 at the position of mass m_1

$$R_{COM} = \frac{(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots m_N\vec{r}_N)}{M}$$

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad x_1 = 0, \quad x_2 = L$$

$$x_{COM} = \frac{m_2}{m_1 + m_2} L$$

- Now we can calculate the moment of inertia
  - the rotation axis goes through the centre of mass
  - When we calculate the positions of the masses for the moment of inertia, we calculate them relative to the rotation axis.



$$x_{COM} = \frac{m_2}{m_1 + m_2} L$$

Distance from CM to m_1 is

$$r_1 = X = \frac{m_2 L}{M}$$

Distance from CM to m_2 is

$$r_2 = L - X = L - \frac{m_2 L}{M}$$

- Now calculate the moment of inertia

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

$$I = m_1\left(\frac{m_2 L}{M}\right)^2 + m_2\left(L - \frac{m_2 L}{M}\right)^2$$

$$I = m_1\left(\frac{m_2 L}{M}\right)^2 + m_2\left(\frac{LM - m_2 L}{M}\right)^2$$

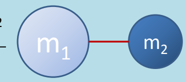
$$I = m_1\left(\frac{m_2 L}{M}\right)^2 + m_2\left(\frac{m_1 L}{M}\right)^2$$

$$I = \frac{m_1 m_2^2 L + m_2 m_1^2 L}{M}$$

$$I = \frac{(m_1 + m_2) m_1 m_2 L^2}{(m_1 + m_2)^2}$$

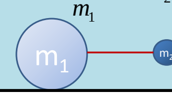
$$I = \frac{m_1 m_2 L^2}{m_1 + m_2}$$

- The final result: $I = \frac{m_1 m_2 L^2}{m_1 + m_2}$
- For equal masses $I = \frac{mL^2}{2}$



- For very unequal masses $I \approx \frac{m_1 m_2 L^2}{m_1} = m_2 L^2$

If $m_1 >> m_2$



Rotational Inertia for a Continuous Mass Distribution

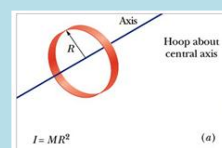
- This is where we integrate over the mass of the object

$$I = \int r^2 dm$$

- The integral takes into account the shape of the object

Hoop or Thin Walled Cylinder

- Cylindrical shell
- Rotational axis about the central axis
- “Thin” means consider all the mass at the same distance R from the axis of rotation

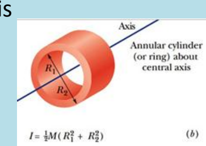


$$I = MR^2$$

The Annular Cylinder

- The annular cylinder, with a finite thickness is a more reasonable physical model
- We consider this cylinder to consist of many concentric shells, each with radius r and thickness dr.
- The volume of each cylinder is

$$dV = 2\pi rhdr$$



The Annular Cylinder

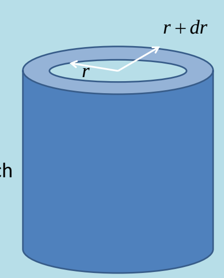
- The mass of each cylinder is

$$dm = \rho dV$$

$$dm = \rho \times 2\pi rhdr$$

- The moment of inertia of each cylinder is

$$dI = r^2 dm$$



$$dI = r^2 dm$$

$$dI = r^2 \times 2\pi\rho h r dr$$

$$dI = 2\pi\rho h r^3 dr$$

We now use an integration, from r = R_1 to r = R_2 (the outer radius of the solid cylinder), to find the total moment of inertia

$$I = \int_{R_1}^{R_2} 2\pi\rho h r^3 dr$$

$$I = 2\pi\rho h \int_{R_1}^{R_2} r^3 dr$$

Constant, can be taken outside the integral

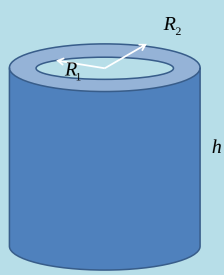
$$I = 2\pi\rho h \left[\frac{r^4}{4}\right]_{R_1}^{R_2}$$

$$I = \pi\rho h \left[\frac{R_2^4}{2} - \frac{R_1^4}{2}\right]$$

We have an additional piece of information:

$$\rho = \frac{mass}{volume} = \frac{M}{V}$$

$$V = \pi R_2^2 h - \pi R_1^2 h$$



$$I = \pi\rho h \left[\frac{R_2^4}{2} - \frac{R_1^4}{2}\right]$$

Replace the density here with the expressions

$$\rho = \frac{M}{V}$$

$$V = \pi R_2^2 h - \pi R_1^2 h$$

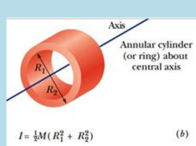
$$I = \frac{\pi M h}{\pi h \left[R_2^2 - R_1^2\right]} \left[\frac{R_2^4}{2} - \frac{R_1^4}{2}\right]$$

$$I = \frac{1}{2} \frac{M}{\left[R_2^2 - R_1^2\right]} \left[R_2^4 - R_1^4\right]$$

$$(a+b)(a-b) = a^2 - b^2$$

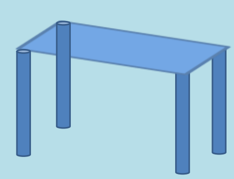
$$I = \frac{1}{2} \frac{M}{\left[R_2^2 - R_1^2\right]} \left[R_2^2 - R_1^2\right]\left[R_2^2 + R_1^2\right]$$

$$I = \frac{1}{2} M \left[R_2^2 + R_1^2\right]$$



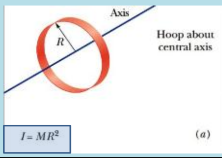
- Since moments of inertia are integrals, we can also break down complex shapes into easier combinations of simple geometries, where the moments of inertia are known

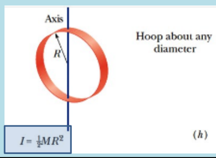
Table could be a rectangular slab and 4 cylinders



Moment of Inertia and the Rotation Axis

- The value of the moment of inertia depends on the distance of the mass away from the axis of rotation, and is different for different rotation axes (unlike mass which is always constant)

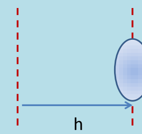




Parallel Axis Theorem

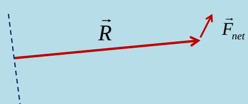
- There is also a simple relationship between moment of inertia of a solid when rotated about an axis which is parallel to one through the centre of mass
- This is known as the parallel axis theorem

$$I = I_{com} + Mh^2$$



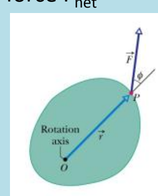
Torque

- The torque on a body is the agent which causes the object to change its rotational motion
- It is the equivalent to force in a linear system.



Torque → $\vec{\tau}_{net} = (\vec{r} \times \vec{F}_{net})$ ←

- The torque is the vector product (cross product) of the displacement vector r from the axis to the point of application of a net force F_{net}



The torque due to this force causes rotation around this axis (which extends out toward you).

$$\vec{\tau}_{net} = (\vec{r} \times \vec{F}_{net})$$

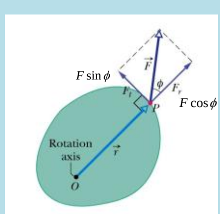
$$|\tau_{net}| = r(F \sin\phi) = F(r \sin\phi)$$

From the definition of vector(cross) product of tow vectors

- The tangential force component is F sinϕ

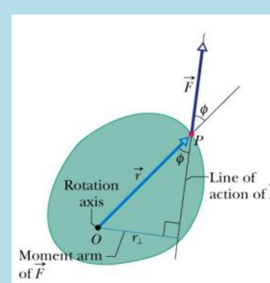
$$|\tau_{net}| = r(F \sin\phi)$$

- Torque is the product of the component of the force perpendicular to the moment arm (the perpendicular distance from the point of application to the force from the axis)



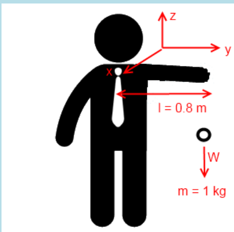
But actually only the tangential component of the force causes the rotation.

- Or torque can be thought of as the component of the moment arm distance perpendicular to the force.
  - All of these definitions are equivalent
  - We use whichever definition is easiest to evaluate, given the nature of the problem to be sold.



You calculate the same torque by using this moment arm distance and the full force magnitude.

Torque Calculation

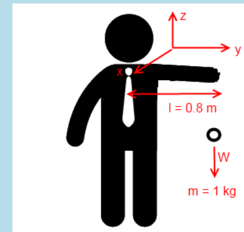


- Stone, mass m = 1 kg
- Held in an outstretched arm, length l = 0.8 m
- Find torque from the origin (shoulder)

Lever Arm Method

$l = 0.8m$

$\vec{F} = m\vec{g}$



$\tau = lmg$

$\tau = (0.8\,\mathrm{m})(1\mathrm{kg})(9.81\,\mathrm{m/s^2})$

$\tau = 8\,\mathrm{n.m}$

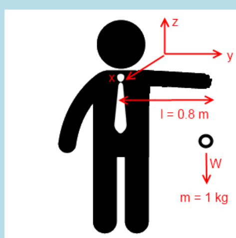
Direction of torque is clockwise:
Torque has a negative sign

Vector Method

Torque $\vec{\tau} = \vec{r} \times \vec{F}$

$\vec{F} = -mg\mathbf{k}$

$\vec{r} = l\,\mathbf{j}$



From the definition of the cross product

$\vec{\tau} = \vec{r} \times \vec{F} =$

$(r_y F_z - r_z F_y)\hat{i} +$

$(r_z F_x - r_x F_z)\hat{j} +$

$(r_x F_y - r_y F_x)\hat{k}$

Force only has a z component

$\vec{F} = -mg\hat{k}$

$F_z = -mg$

Displacement only has a y component

$\vec{r} = l\,\hat{j}$

$r_y = l$

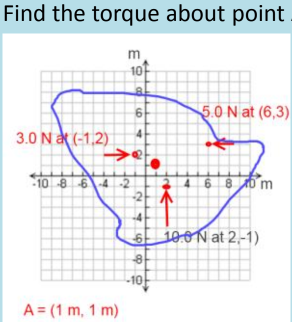
$\tau = \vec{r} \times \vec{F} = r_y F_z \mathbf{i}$

$\tau = \vec{r} \times \vec{F} = -lmg_z i$

Same result as the lever arm method

Torque Example: 3 Forces Acting

- Find the torque about point A

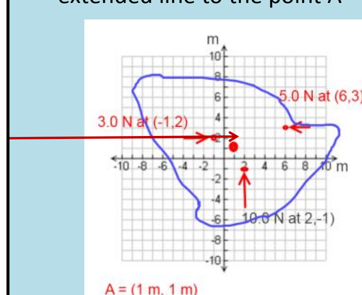


$\vec{F}_1 = +(3.0\,\mathrm{N})\mathbf{i}$

$\vec{F}_2 = -(5.0\,\mathrm{N})\mathbf{i}$

$\vec{F}_3 = +(10.0\,\mathrm{N})\mathbf{j}$

- Extend the lines of the force vectors
- Find the perpendicular distance from the extended line to the point A



$\vec{F}_1 = +(3.0\,\mathrm{N})\mathbf{i}$

$\mathrm{d}_1 = 1.0\,m$

Using the right hand rule, the direction of the torque is into the page (negative z axis)

$\tau_1 = d_1 F_1 \sin 90^0$

$\vec{\tau}_1 = -(3.0\,\mathrm{N.m})\mathbf{k}$

- Do the same for the other two forces
- Extend the lines of the force vectors
- Find the perpendicular distance from the extended line to the point A

$$\vec{F}_2 = -(5.0\ \text{N})\mathbf{i}$$
$$d_2 = 2.0\ m$$

Using the right hand rule, the direction of the torque is out of the page (+ve z axis)

$$\tau_2 = d_2 F_2 \sin 90^0$$
$$\vec{\tau}_2 = +(1.0\times10^1\ \text{N.m})\mathbf{k}$$

- Extend the lines of the force vectors
- Find the perpendicular distance from the extended line to the point A

$$\vec{F}_3 = +(10.0\ \text{N})\mathbf{j}$$
$$d_3 = 1.0\ m$$

Using the right hand rule, the direction of the torque is out of the page (+ve z axis)

$$\tau_3 = d_3 F_3 \sin 90^0$$
$$\vec{\tau}_3 = +(1.0\times10^1\ \text{N.m})\mathbf{k}$$

- The net torque about point A is then

$$\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3$$

$$\vec{\tau}_{net} = -(3.0\ \text{N.m})\mathbf{k} + +(1.0\times10^1\ \text{N.m})\mathbf{k} + (1.0\times10^1\ \text{N.m})\hat{\text{k}}$$

$$\vec{\tau}_{net} = +(17\ \text{N.m})\vec{\text{k}}$$

- Using the right hand rule, this means that the direction of rotation is counter clockwise.
- Don't confuse the rotation direction with the direction of the torque vector, which is out of the screen/page

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Newton's Second Law for Rotational Motion

- Newton's Second Law for Rotational Motion becomes

$$\vec{\tau}_{net} = I\vec{\alpha}$$

Eqn. 10.42

Torque

Moment of inertia

Angular acceleration

- Which compares directly with $\vec{F}_{net} = m\vec{a}$

$$|\tau_{net}| = r(F\sin\phi)$$

- The SI unit of Torque is the Newton.metre, N.m
- Note that it is ALWAYS written as Newton.metre or N.m to identify it as a rotational unit.
- Work may be expressed in either Joules or N.m.
  - (I suggest always using Joules to avoid confusion)
- Torque and work are not equivalent quantities, even though they have the same units

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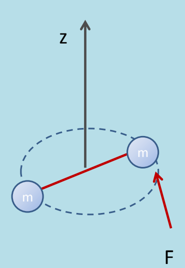
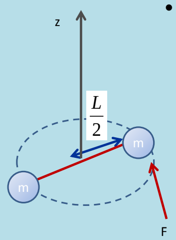
Example: Find the Torque and Angular Acceleration



- The rod connecting the two masses m is assumed to be mass-less and has length L
- A tangential force F is applied to one sphere
- The spheres can be approximated as point masses
- Assume m = 0.10 kg, L = 25 cm, F = 2.0 N

Find The Torque



- The force is tangential (perpendicular) to the line from the centre of rotation to the mass being pushed

$$\tau = Fr\sin\phi$$

$$\tau = Fr\sin 90^{o}$$

$$\tau = \frac{FL}{2}$$

- We can now calculate the torque on the system
- m = 0.10 kg, L = 25 cm, F = 2.0 N

$$\tau = \frac{FL}{2}$$

$$\tau = \frac{(2.0\ \text{N})(0.25\ \text{m})}{2}$$

$$\tau = 0.25\ \text{N.m}$$

- To find the angular acceleration, we will need to use Newton's Second Law

$$\vec{\tau} = I\vec{\alpha}$$

We just calculated this

We can calculate the rotational inertia from the mass distribution

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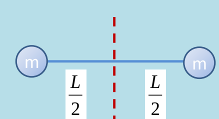
Find the Moment of Inertia

- Treat the masses as point masses.

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

$$I = m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{2}$$



Apply Newton's Second Law:

$$\vec{\tau} = I\vec{\alpha}$$

$$F\left(\frac{L}{2}\right) = \frac{mL^2}{2}\alpha$$

$$\alpha = \frac{F}{mL}$$

- Drop the vector notation here and use magnitudes,
- We can do this since torque and angular acceleration have the same direction, determined by the right hand rule
- The direction is upwards, along the positive z axis

- Calculate the angular acceleration

$$\alpha = \frac{F}{mL}$$

$$\alpha = \frac{2.0\ \text{N}}{(0.10\ \text{kg})(0.25\ \text{m})} = 8.0\times 10^{1}\ \text{rad/s}^{2}$$

- Now we'll extend this problem.
- Assume the contact time of the object causing the force was 0.20 seconds.
  a) Calculate the final angular velocity, assuming that the initial velocity was zero.
  b) Calculate the angular displacement

$$\omega = \omega_0 + \alpha t$$

$$\omega = 0 + (8.0 \times 10^1 \text{ rad/s}^2)(0.20 s) = 16 \text{ rad/s}$$

- Now the angular displacement

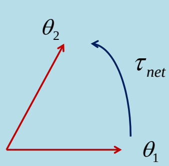
$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$(16 \text{ rad/s})^2 = 0 + 2(8.0 \times 10^1 \text{ rad/s}^2)\Delta\theta$$

$$\Delta\theta = \frac{(16 \text{ rad/s})^2}{2(8.0 \times 10^1 \text{ rad/s}^2)} = 0.10 \text{ radians}$$

Work and Rotational Kinetic Energy

- The work done by a constant torque can be defined in terms of the angular displacement
- If the torque produces an angular displacement Δθ, then the work done is



$$W = \tau\Delta\theta$$

Equation 10.54

- If the torque is variable, then the work done can be found by integration:

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$$

Equation 10-53

Rotational Work	Translational Work
$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$	$W = \int_{x_i}^{x_f} F dx$
$W = \tau\Delta\theta$	$W = Fx\cos\phi$

- In the example we have been calculating, we know the torque 0.25 N.m (assumed constant)
- And the angular displacement, 0.10 radians

$$W = \tau\Delta\theta$$

$$W = (0.25 \text{ N.m})(0.1 \text{ radians}) = 0.025 \text{ J}$$

Work-Kinetic Energy Theorem

- The rotational work done on a system is equal to the change in the rotational kinetic energy of the system

$$W = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

Eqn. 10-52

- This is exactly analogous to the work-energy theorem for translational motion

$$W_{tr} = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

- We can use our example to calculate the change in KE, from the work done to make it move from rest to the final angular speed

$$W = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

$$I = \frac{mL^2}{2} = \frac{(0.10\,\text{kg})(0.20\,\text{m})^2}{2} = 2.0\times10^{-3}\ \text{kg.m}^2$$

$$W = 0.25\,\text{J}$$

$$\omega_i = 0\ \text{rad/s}$$

$$I = 2.0\times10^{-3}\ \text{kg.m}^2$$

$$W = 0.25\,\text{J}$$

$$\omega_i = 0\ \text{rad/s}$$

$$W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 \quad \Rightarrow \quad \omega_f = \sqrt{\frac{2W}{I}}$$

$$\omega_f = \sqrt{\frac{2\times0.25\,\text{J}}{2.0\times10^{-3}\ \text{kg.m}^2}} = 15.8\ \text{rad/s} = 16\ \text{rad/s to 2 s.f.}$$

- This is consistent with our calculation of the final angular velocity from the kinematics

Rotational Power

- The Rotational Power is the rate at which work is done on the system (which is the rate of change of energy of the system)

$$P = \frac{dW}{dt}$$

- For systems with constant angular speed and torque

$$P = \tau\omega$$