

**RYERSON UNIVERSITY**  
**DEPARTMENT OF MATHEMATICS**  
**MTH 510 - Numerical Analysis**

**Midterm Test - Fall 2012**

Last Name (print): \_\_\_\_\_ SOLUTIONS  
 First Name (print): \_\_\_\_\_ IVORY  
 Student ID Number: \_\_\_\_\_  
 Signature: \_\_\_\_\_  
 Course: \_\_\_\_\_

Course	Section	Lab day/time	Location	Instructor
MTH 510	1	Mon 11am-noon	LIB 393	Dr. Ilie
MTH 510	2	Mon 8-9am	LIB 393	Dr. Ilie
MTH 510	3	Tues 1-2pm	LIB 393	Dr. Ilie
MTH 510	4	Tues 3-4pm	LIB 393	Dr. Homayouni
MTH 510	5	Mon 3-4pm	LIB 393	Dr. Homayouni
MTH 510	6	Fri 2-3pm	LIB 393	Dr. Homayouni
MTH 510	7	Tues 2-3pm	KHE 137	Dr. Homayouni

Date: October 26, 2012, 4:20 pm

Time Allowed: 1.5 hours

**INSTRUCTIONS:**

- Verify that the test contains all 8 pages, including this cover page.
- Use a pen or pencil and write legibly for full marks.

The examination has two parts:

**Part A consists of full-solution questions** with the mark for each full solution as indicated. Answer all questions in the space provided. Clearly explain your methods, and show all relevant steps in the solution. An answer consisting only of the final result will be given little or no credit.

- Part B contains Multiple Choice questions.** Clearly write your answer in the space provided. No part marks will be given and no marks will be deducted for incorrect answers. If more than one answer is given, a mark of zero will be assigned to that question.

- This is a closed-book test.
- When not specified, six significant digits of accuracy is sufficient.
- The last page is for rough work.  
DO NOT SEPARATE THE PAGES
- Permitted Aids:
  - One handwritten 8.5 x 11 inch Formula sheet (both sides),
  - Non-programmable scientific calculators.

For instructor's use only.

Question(s)	Value	Mark
1	8	
2	12	
3	8	
4	6	
5	8	
6-9	8	
<b>Total</b>	50	

1. (8 marks) Use Gauss-Seidel iteration to solve

$$\begin{aligned} 5x_1 + x_2 + 2x_3 &= 10 \\ -3x_1 + 9x_2 + 4x_3 &= -14 \\ x_1 + 2x_2 - 7x_3 &= -33 \end{aligned}$$

Do three iterations starting with initial guess  $[0, 0, 0]^T$  and give the absolute value of the approximate percent relative error at each iteration.

Solution: Gauss-Seidel method:

$$\text{for } k=0, 1, 2, \dots \begin{cases} x_1^{(k+1)} = \frac{1}{5} [10 - x_2^{(k)} - 2x_3^{(k)}] \\ x_2^{(k+1)} = \frac{1}{9} [-14 + 3x_1^{(k+1)} - 4x_3^{(k)}] \\ x_3^{(k+1)} = -\frac{1}{7} [-33 - x_1^{(k+1)} - 2x_2^{(k+1)}] \end{cases} \quad (2)$$

First iteration  $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

$$x_1^{(1)} = \frac{1}{5} [10 - x_2^{(0)} - 2x_3^{(0)}] = \frac{1}{5} [10 - 0 - 0] = \boxed{2}$$

$$x_2^{(1)} = \frac{1}{9} [-14 + 3(2) - 4(0)] = -\frac{8}{9} = \boxed{-0.888888}$$

$$x_3^{(1)} = -\frac{1}{7} [-33 - 2 - 2(-\frac{8}{9})] = \frac{299}{63} = \boxed{4.746032}$$

Approx % rel error

$$\varepsilon_{a,1}^{(1)} = \left| \frac{2-0}{2} \right| \cdot 100\% = 100\%$$

$$\varepsilon_{a,2}^{(1)} = \left| \frac{-\frac{8}{9} - 0}{-\frac{8}{9}} \right| \cdot 100\% = 100\%$$

$$\varepsilon_{a,3}^{(1)} = \left| \frac{\frac{299}{63} - 0}{\frac{299}{63}} \right| \cdot 100\% = 100\%$$

Second iteration

$$x_1^{(2)} = \frac{1}{5} [10 - (-\frac{8}{9}) - 2(\frac{299}{63})] = \boxed{0.279365}$$

$$x_2^{(2)} = \frac{1}{9} [-14 + 3(0.279365) - 4(\frac{299}{63})] = \boxed{-3.571781}$$

$$x_3^{(2)} = -\frac{1}{7} [-33 - 0.279365 - 2(-3.571781)] = \boxed{3.733686}$$

Approximate percent relative error:

$$\varepsilon_{a,1}^{(2)} = \left| \frac{0.279365 - 2}{0.279365} \right| \cdot 100\% = 615.9\%$$

$$\varepsilon_{a,2}^{(2)} = \left| \frac{-3.571781 + 0.888888}{-3.571781} \right| \cdot 100\% = 75.11\%$$

$$\varepsilon_{a,3}^{(2)} = \left| \frac{3.733686 - 4.746032}{3.733686} \right| \cdot 100\% = 27.11\%$$

Q1 - continued

Third iteration

$$x_1^{(3)} = \frac{1}{5} [10 - x_1^{(2)} - 2x_2^{(2)}] = \boxed{1.220882}$$

$$(2) \quad x_2^{(3)} = \frac{1}{9} [-14 + 3x_1^{(3)} - 4x_3^{(2)}] = \boxed{-2.808011}$$

$$x_3^{(3)} = -\frac{1}{7} [-33 - x_1^{(3)} - 2x_2^{(3)}] = \boxed{4.086409}$$

Approx. percent rel. errors.

$$\epsilon_{a,1}^{(3)} = \left| \frac{x_1^{(3)} - x_1^{(2)}}{x_1^{(3)}} \right| \cdot 100\% = 77.11\%$$

$$\epsilon_{a,2}^{(3)} = \left| \frac{x_2^{(3)} - x_2^{(2)}}{x_2^{(3)}} \right| \cdot 100\% = 27.19\%$$

$$\epsilon_{a,3}^{(3)} = \left| \frac{x_3^{(3)} - x_2^{(3)}}{x_3^{(3)}} \right| \cdot 100\% = 8.63\%$$

2. Let  $A$  be the matrix

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) (6 marks) Find the LU-decomposition (without pivoting) of the matrix  $A$  and write the resulting matrices in the space below:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) (4 marks) Use the LU decomposition of  $A$  to find the solution of  $Ax = b$  where  $b = [2, -6, -1]^T$ .

(c) (2 marks) Use the LU-factorization to compute the determinant of  $A$ .

Solution:

(a)  $A = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$   $\xrightarrow{\textcircled{2}}$

$R_2 \leftarrow R_2 - \frac{4}{4}R_1 = R_2 - R_1$   
 $R_3 \leftarrow R_3 - \frac{2}{4}R_1 = R_3 - \frac{1}{2}R_1$

$\rightarrow \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \textcircled{1}$   $\xrightarrow{R_3 \leftrightarrow R_3 - \frac{1}{2}R_2}$   $\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U \textcircled{1}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \textcircled{2}$

(b) Solve  $Ld = b$   $\textcircled{1}$

by forward subst.  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix}$   $\textcircled{1}$

$\Rightarrow d_1 = 2$   $2 + d_2 = -6 \Rightarrow d_2 = -8$   $\rightarrow d = \begin{bmatrix} 2 \\ -8 \\ 2 \end{bmatrix} \textcircled{1}$

$\frac{1}{2}(2) + \frac{1}{2}(-8) + d_3 = -1 \Rightarrow d_3 = 2$

Q2b cont

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Solve  $Ux = d$  (1)

by back ward subst  $\Rightarrow$

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 2 \end{bmatrix}$$

$$\Rightarrow 2x_3 = 2 \Rightarrow x_3 = 1$$

$$2x_2 + 2(1) = -8 \Rightarrow x_2 = -5$$

$$4x_1 + 2(-5) = 2 \Rightarrow x_1 = 3$$

$$\Rightarrow x = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} \text{ (1)}$$

$$\textcircled{c} \det(A) = \det(L \cdot U) = (\det L)(\det U) = (1 \times 1 \times 1) \times (4 \times 2 \times 2)$$

$$\Rightarrow \det(A) = 16 \quad \textcircled{2}$$

3. (8 marks) Use Gaussian elimination with partial pivoting and backward substitution to find the solution of

$$\begin{bmatrix} 2 & 6 & 10 \\ 1 & 3 & 3 \\ 3 & 14 & 28 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -8 \end{bmatrix}$$

Solution : Augmented matrix

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 2 & 6 & 10 & 0 \\ 1 & 3 & 3 & 2 \\ 3 & 14 & 28 & -8 \end{array} \right] \xrightarrow{\text{(pivoting) swap } R_1 \& R_3} \left[ \begin{array}{ccc|c} 3 & 14 & 28 & -8 \\ 1 & 3 & 3 & 2 \\ 2 & 6 & 10 & 0 \end{array} \right] \begin{array}{l} R \\ R \\ R \end{array}$$

pivot ①

$$\rightarrow \left[ \begin{array}{ccc|c} 3 & 14 & 28 & -8 \\ 0 & -5/3 & -19/3 & 14/3 \\ 0 & -10/3 & -26/3 & 16/3 \end{array} \right] \begin{array}{l} \text{pivoting} \\ \text{swap } R_2 \& R_3 \end{array} \left[ \begin{array}{ccc|c} 3 & 14 & 28 & -8 \\ 0 & -10/3 & -26/3 & 16/3 \\ 0 & -5/3 & -19/3 & 14/3 \end{array} \right] \begin{array}{l} R \\ R \\ R \end{array}$$

pivot ②

$\left. \begin{array}{l} R_2 \leftrightarrow R_2 - \frac{1}{3}R_1 \\ R_3 \leftrightarrow R_3 - \frac{2}{3}R_1 \end{array} \right\}$

$$\rightarrow \left[ \begin{array}{ccc|c} 3 & 14 & 28 & -8 \\ 0 & -10/3 & -26/3 & 16/3 \\ 0 & -5/3 & -19/3 & 14/3 \end{array} \right] \begin{array}{l} R_2 \\ R_3 \end{array} \begin{array}{l} R_2 \\ R_3 \leftrightarrow R_3 - \frac{1}{2}R_2 \end{array} \left[ \begin{array}{ccc|c} 3 & 14 & 28 & -8 \\ 0 & -10/3 & -26/3 & 16/3 \\ 0 & -2 & -2 & 2 \end{array} \right] \begin{array}{l} R \\ R \\ R \end{array}$$

$\Rightarrow$  Solve by backward substitution ②

$$-2x_3 = 2 \Rightarrow x_3 = -1$$

$$-10/3 x_2 - 26/3(-1) = 16/3 \Rightarrow x_2 = 1$$

$$3x_1 + 14(1) + 28(-1) = -8 \Rightarrow x_1 = 2$$

Solution is  $x = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$  ①

4 (6 marks) Use the secant method with  $x_0 = 1$  and  $x_1 = 2$  to approximate the root of

$$f(x) = x^3 - 2x - 2$$

until the approximate percent relative error satisfies the criterion for 1 significant digit.

Solution: Secant method:  $x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$  (1)

Initial guesses  $x_0 = 1, x_1 = 2$ .

$i = 1, 2, 3, \dots$

Stopping criterion:  $|\epsilon_a| < \epsilon_s = 0.5 \times 10^{2-1} \% = 5\%$  (1)

$i$	$x_{i-1}$	$x_i$	$f(x_{i-1})$	$f(x_i)$	$ \epsilon_a  = \left  \frac{x_i - x_{i-1}}{x_i} \right  \%$
1	1	$x_1 = 2$	-3	2	$\left  \frac{2-1}{2} \right  \cdot 100\% = 5\%$
2	2	$x_2 = \frac{8}{5} = 1.6$	2	-1.104000	$ \epsilon_a  = 24.9999\%$
3	1.6	$x_3 = 1.742268$	-1.104000	-0.195885	$ \epsilon_a  = 8.16\%$
4	1.742268	$x_4 = 1.772956$	-0.195885		$ \epsilon_a  = 1.73\%$

Sample calculation

$$i = 1 \Rightarrow x_2 = x_1 - \frac{f(x_1)(x_0 - x_1)}{f(x_0) - f(x_1)}$$

$$\Rightarrow x_2 = 2 - \frac{f(2)(1-2)}{f(1)-f(2)} = 1.6$$

Approx percent relative error

$$|\epsilon_a| = \left| \frac{1.6 - 2}{1.6} \right| \cdot 100\% = 24.9999\%$$

Answer  $x_3 = 1.7442268$

has at least 1 significant digit

5. (a) (5 marks) Using 3 iterations of the Bisection method, estimate the root of

$$f(x) = \cos^2 x + 6 - x$$

in the interval  $[6, 7]$ . What is the maximum error made in the estimate for the root of  $f(x)$ ?

(b) (3 marks) How many iterations would be required for the true error to be less than  $10^{-8}$ ?

Solution

Bisection method:  $x_r = \frac{x_l + x_u}{2}$

a)	i	$x_l$	$x_r$	$x_u$	$f(x_l)$	$f(x_r)$	$f(x_u)$	$\max \epsilon_t $
	1	6	6.5	7	+0.92	+0.45	-0.43	$\frac{7-6}{2} = 0.5$
3	2	6.5	6.75	7	+0.45	+0.04	-0.43	$\frac{7-6.5}{2} = 0.25$
	3	6.75	6.875	7	+0.04		-0.43	$\frac{7-6.75}{2} = 0.125$

After 3 iteration of the bisection method

$$x_r = 6.875 \quad (1)$$

and the maximum error of the estimate for the root is

$$\max|\epsilon_t| \leq 0.125 \rightarrow (1)$$

b)  $E_{a,d} = 10^{-8}$

$$(2) \quad n = \log_2 \left( \frac{7-6}{10^{-8}} \right) = \frac{\log(7-6) - \log 10^{-8}}{\log 2} = 26.5754 \approx 27$$

$n = 27$  iterations are needed for  
(1) the true error  $\leq 10^{-8}$

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**Part B - Multiple Choice Questions**

6. (2 marks) Which of the following commands in MATLAB can be used to solve the system  $Ax = b$ , where  $A$  is a matrix and  $b$  a vector

- (A) `x=inv(A).*b`
- (B) `x=inverse(A)*b`
- (C) `x=A^{-1}*b`
- (D) None of (A)-(C) are true.

ANSWER: 6. D

7. (2 marks) The following commands have been entered in MATLAB:

```
A=[5, 6, 9; -10, 2, 7; 1, -5, 15]
c=norm(A,Inf)
```

- (A) The value obtained is  $c = 31$ .
- (B) The value obtained is  $c = 21$ .
- (C) The value obtained is  $c = 18.8578$ .
- (D) None of (A)-(C) are true.

ANSWER: 7. B

8. (2 marks) To plot  $y = \cos(x) + \exp(x)x$  for  $x \in [-4, 4]$  which of the following commands gives an **error message** in MATLAB:

- (A) `x=linspace(-4,4); f=@(x) cos(x)+exp(x)*x;`  
`plot(x,f(x))`
- (B) `x=[-4:0.1:4]; y=cos(x)+exp(x).*x;`  
`plot(x,y)`
- (C) `x=linspace(-4,4); y=@(x) cos(x)+exp(x).*x;`  
`plot(x, y(x))`
- (D) None of (A)-(C) are true.

ANSWER: 8. A

9. (2 marks) Entering the commands in MATLAB:

```
p=[1, -5, 4];
x=roots(p)
```

will give the result

- (A)  $x = [1; 0.25]$ .
- (B)  $x = [1; 0; -21; 20]$ .
- (C)  $x = [4; 1]$ .
- (D) None of (A)-(C) are true.

ANSWER: 9. C