RYERSON UNIVERSITY

DEPARTMENT OF MATHEMATICS

MTH 510 - Numerical Analysis

Midterm Test - Fall 2012

Last Name (print): SOLUTIONS
First Name (print): WHITE
Student ID Number:
Signature:
Course:

Course	Section	Lab day/time	Location	Instructor
MTH 510	1	Mon 11am-noon	LIB 393	Dr. Ilie
MTH 510	2	Mon 8-9am	LIB 393	Dr. Ilie
MTH 510	3	Tues 1-2pm	LIB 393	Dr. Ilie
MTH 510	4	Tues 3-4pm	LIB 393	Dr. Homayouni
MTH 510	5	Mon 3-4pm	LIB 393	Dr. Homayouni
MTH 510	6	Fri 2-3pm	LIB 393	Dr. Homayouni
MTH 510	7	Tues 2-3pm	KHE 137	Dr. Homayouni

Date: October 26, 2012, 4:20 pm **Time Allowed: 1.5 hours**

INSTRUCTIONS:

1. Verify that the test contains all 8 pages, including this cover page.

2. Use a pen or pencil and write legibly for full marks.

3. The examination has two parts:
Part A consists of full-solution questions with the mark for each full solution as indicated. Answer all questions in the space provided. Clearly explain your methods, and show all relevant steps in the solution. An answer consisting only of the final result will be given little or no credit.
Part B contains Multiple Choice questions. Clearly write your answer in the space provided. No part marks will be given and no marks will be deducted for incorrect answers. If more than one answer is given, a mark of zero will be assigned to that question.

4. This is a closed-book test.

5. When not specified, six significant digits of accuracy is sufficient.

6. The last page is for rough work.
DO NOT SEPARATE THE PAGES

7. Permitted Aids:
i) One handwritten 8.5 x 11 inch Formula sheet (both sides),
ii) Non-programmable scientific calculators.

For instructor's use only.

Question(s)	Value	Mark
1	8	
2	6	
3	8	
4	12	
5	8	
6-9	8	
Total	50	

1. (a) (5 marks) Using 3 iterations of the Bisection method, estimate the root of

$$f(x) = \cos^2 x + 6 - x$$

in the interval $[6, 7]$. What is the maximum error made in the estimate for the root of $f(x)$?

(b) (3 marks) How many iterations would be required for the true error to be less than 10^{-8}?

Solution

Bisection method: $x_r = \dfrac{x_\ell + x_u}{2}$

a) 3

i	x_ℓ	x_r	x_u	$f(x_\ell)$	$f(x_r)$	$f(x_u)$	$\max\lvert\varepsilon_t\rvert$
1	6	$\boxed{6.5}$	7	$+0.92$	$+0.45$	-0.43	$\frac{7-6}{2} = ($
2	6.5	$\boxed{6.75}$	7	$+0.45$	$+0.04$	-0.43	$\frac{7-6.5}{2} = 0.25$
3	6.75	$\boxed{6.875}$	7	$+0.04$		-0.43	$\frac{7-6.75}{2} = 0.12$

After 3 iteration of the bisection method

$$\boxed{x_r = 6.875} \quad 1$$

and the maximum error of the estimate for the root is

$$\max|\varepsilon_t| \leq 0.125 \rightarrow 1$$

b) 2

$$E_{a,d} = 10^{-8}$$

$$n = \log_2\left(\frac{7-6}{10^{-8}}\right) = \frac{\log(7-6) - \log 10^{-8}}{\log 2} = 26.5754 \simeq 27$$

$\boxed{n = 27}$ iterations are needed for the true error $\leq 10^{-8}$

1

2. (6 marks) Use the secant method with $x_0 = 1$ and $x_1 = 2$ to approximate the root of

$$f(x) = x^3 - 2x - 2$$

until the approximate percent relative error satisfies the criterion for 1 significant digit.

Solution: Secant method: $$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \quad (1$$

$i = 1, 2, 3, \ldots$

Initial guesses $x_0 = 1$, $x_1 = 2$.

Stopping criterion: $|\varepsilon_a| < \varepsilon_s = 0.5 \times 10^{2-1}\,\% = 5\%$ (1

i	x_{i-1}	x_i	$f(x_{i-1})$	$f(x_i)$	$\lvert\varepsilon_a\rvert = \left\lvert\frac{x_i - x_{i-1}}{x_i}\right\rvert \%$
1	1	$x_1 = 2$	-3	2	$\left\lvert\frac{2-1}{2}\right\rvert \cdot 100\% = 5$
2	2	$x_2 = \frac{8}{5} = 1.6$	2	-1.104000	$\lvert\varepsilon_a\rvert = 24.999\%$
3	1.6	$x_3 = 1.742268$	-1.104000	-0.195885	$\lvert\varepsilon_a\rvert = 8.16\%$
4	1.742268	$\boxed{1.772956}$	-0.195885		$\lvert\varepsilon_a\rvert = 1.73\%$

4

$< 5\%$
STOP

Sample calculation

$$i = 1 \Rightarrow x_2 = x_1 - f(x_1)\frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$$

$$\Rightarrow x_2 = 2 - f(2)\frac{(1-2)}{f(1) - f(2)} = 1.6$$

Approx percent relative error

$$|\varepsilon_a| = \left|\frac{1.6 - 2}{1.6}\right| \cdot 100\% = 24.9999\%$$

Answer $\boxed{x_3 = 1.742268}$ has at least 1 significant digit

3. (8 marks) Use Gaussian elimination with partial pivoting and backward substitution to find the solution of

$$\begin{bmatrix} 2 & 6 & 10 \\ 1 & 3 & 3 \\ 3 & 14 & 28 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \\ -8 \end{Bmatrix}$$

Solution : Augmented matrix

pivot 1

(pivoting)

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 6 & 10 & 0 \\ 1 & 3 & 3 & 2 \\ 3 & 14 & 28 & -8 \end{array}\right] \quad \text{swap } R_1 \& R_3 \longrightarrow \left[\begin{array}{ccc|c} \boxed{3} & 14 & 28 & -8 \\ 1 & 3 & 3 & 2 \\ 2 & 6 & 10 & 0 \end{array}\right] \begin{array}{c} R \\ R \\ R \end{array}$$

2

$$\begin{cases} R_2 \leftarrow R_2 - \frac{1}{3} \\ R_3 \leftarrow R_3 - \frac{2}{3} \end{cases}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 3 & 14 & 28 & -8 \\ 0 & -5/3 & -19/3 & 14/3 \\ 0 & -10/3 & -26/3 & 16/3 \end{array}\right] \quad \text{pivoting swap } R_2 \& R_3$$

1

$$\longrightarrow \left[\begin{array}{ccc|c} 3 & 14 & 28 & -8 \\ 0 & -10/3 & -26/3 & 16/3 \\ 0 & -5/3 & -19/3 & 14/3 \end{array}\right] \begin{array}{c} \\ R_2 \\ R_3 \end{array} \quad R_3 \leftarrow R_3 - \tfrac{1}{2} R_2 \longrightarrow \left[\begin{array}{ccc|c} 3 & 14 & 28 & - \\ 0 & -10/3 & -26/3 & 16/3 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

1

$\Rightarrow$ Solve by backward substitution 2

$$-2x_3 = 2 \Rightarrow x_3 = -1$$

$$-10/3\, x_2 - 26/3(-1) = 16/3 \Rightarrow x_2 = 1$$

$$3x_1 + 14(1) + 28(-1) = -8 \Rightarrow x_1 = 2$$

Solution is $x = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ 1

4. Let A be the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) (6 marks) Find the LU-decomposition (without pivoting) of the matrix A and write the resulting matrices in the space below:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) (4 marks) Use the LU decomposition of A to find the solution of $Ax = b$ where $b = [2, -6, -1]^T$.

(c) (2 marks) Use the LU-factorization to compute the determinant of A.

Solution:

(a) $A = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$ $\longrightarrow$ (2)

$R_2 \leftarrow R_2 - \frac{4}{4} R_1 = R_2 - R_1$

$R_3 \leftarrow R_3 - \frac{2}{4} R_1 = R_3 - \frac{1}{2} R_1$

$\longrightarrow \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \\ R_2 \\ R_3 \end{matrix}$ (1) $\xrightarrow{R_3 \leftarrow R_3 - \frac{1}{2} R_2} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U$ (1)

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}$ (2)

(b) Solve $\boxed{Ld = b}$ (1) by forward subst. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix}$

$\Rightarrow d_1 = 2 \qquad 2 + d_2 = -6 \Rightarrow d_2 = -8 \qquad \Rightarrow d = \begin{bmatrix} 2 \\ -8 \\ 2 \end{bmatrix}$ (1)

$\frac{1}{2}(2) + \frac{1}{2}(-8) + d_3 = -1 \Rightarrow d_3 = 2$

$\boxed{Q4b}$ cont

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Solve $\boxed{Ux=d}$ (1)

by backward subst $\Rightarrow \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 2 \end{bmatrix}$

$\Rightarrow 2x_3 = 2 \Rightarrow x_3 = 1$

$2x_2 + 2(1) = -8 \Rightarrow x_2 = -5$

$4x_1 + 2(-5) = 2 \Rightarrow x_1 = 3$

$\Rightarrow x = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$ (1)

(c) $\det(A) = \det(L \cdot U) = (\det L)(\det U) = (1 \times 1 \times 1) \times (4 \times 2 \times 2)$

$\Rightarrow \det(A) = 16$ (2)

5. (8 marks) Use Gauss-Seidel iteration to solve

$$\begin{aligned} 5x_1 + x_2 + 2x_3 &= 10 \\ -3x_1 + 9x_2 + 4x_3 &= -14 \\ x_1 + 2x_2 - 7x_3 &= -33 \end{aligned}$$

Do three iterations starting with initial guess $[0,0,0]^T$ and give the absolute value of the approximate percent relative error at each iteration.

Solution: Gauss-Seidel method:

for $k = 0, 1, 2, \ldots$

$$\begin{cases} x_1^{(k+1)} = \frac{1}{5}\left[10 - x_2^{(k)} - 2x_3^{(k)}\right] \\ x_2^{(k+1)} = \frac{1}{9}\left[-14 + 3x_1^{(k+1)} - 4x_3^{(k)}\right] \\ x_3^{(k+1)} = -\frac{1}{7}\left[-33 - x_1^{(k+1)} - 2x_2^{(k+1)}\right] \end{cases}$$ (2)

First iteration $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

(2)

$$x_1^{(1)} = \frac{1}{5}\left[10 - x_2^{(0)} - 2x_3^{(0)}\right] = \frac{1}{5}\left[10 - 0 - 0\right] = \boxed{2}$$

$$x_2^{(1)} = \frac{1}{9}\left[-14 + 3(2) - 4(0)\right] = -\frac{8}{9} = \boxed{-0.888888}$$

$$x_3^{(1)} = -\frac{1}{7}\left[-33 - 2 - 2\left(-\frac{8}{9}\right)\right] = \frac{299}{63} = \boxed{4.746032}$$

Approx % rel error

$$\varepsilon_{a,1}^{(1)} = \left|\frac{2-0}{2}\right| \cdot 100\% = 100\%$$

$$\varepsilon_{a,2}^{(1)} = \left|\frac{-\frac{8}{9} - 0}{-\frac{8}{9}}\right| \cdot 100\% = 100\%$$

$$\varepsilon_{a,3}^{(1)} = \left|\frac{\frac{299}{63} - 0}{\frac{299}{63}}\right| \cdot 100\% = 100\%$$

Second iteration

(2)

$$x_1^{(2)} = \frac{1}{5}\left[10 - \left(-\frac{8}{9}\right) - 2\left(\frac{299}{63}\right)\right] = \boxed{0.279365}$$

$$x_2^{(2)} = \frac{1}{9}\left[-14 + 3(0.279365) - 4\left(\frac{299}{63}\right)\right] = \boxed{-3.571781}$$

$$x_3^{(2)} = -\frac{1}{7}\left[-33 - 0.279365 - 2(-3.571781)\right] = \boxed{3.733686}$$

Approximate percent relative error:

$$\varepsilon_{a,1}^{(2)} = \left|\frac{0.279365 - 2}{0.279365}\right| \cdot 100\% = 615.9\%$$

$$\varepsilon_{a,2}^{(2)} = \left|\frac{-3.571781 + 0.888888}{-3.571781}\right| \cdot 100\% = 75.11\%$$

$$\varepsilon_{a,3}^{(2)} = \left|\frac{3.733686 - 4.746032}{3.733686}\right| \cdot 100\% = 27.11\%$$

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Q5 - continued

Third iteration

$$x_1^{(3)} = \frac{1}{5}\left[10 - x_1^{(2)} - 2x_2^{(2)}\right] = \boxed{1.220882}$$

(2) $$x_2^{(3)} = \frac{1}{9}\left[-14 + 3x_1^{(3)} - 4x_3^{(2)}\right] = \boxed{-2.808011}$$

$$x_3^{(3)} = -\frac{1}{7}\left[-33 - x_1^{(3)} - 2x_2^{(3)}\right] = \boxed{4.086409}$$

Approx. percent rel. errors.

$$\varepsilon_{a,1}^{(3)} = \left|\frac{x_1^{(3)} - x_1^{(2)}}{x_1^{(3)}}\right| \cdot 100\% = 77.11\%$$

$$\varepsilon_{a,2}^{(3)} = \left|\frac{x_2^{(3)} - x_2^{(2)}}{x_2^{(3)}}\right| \cdot 100\% = 27.19\%$$

$$\varepsilon_{a,3}^{(3)} = \left|\frac{x_3^{(3)} - x_2^{(3)}}{x_3^{(3)}}\right| \cdot 100\% = 8.63\%$$

Part B - Multiple Choice Questions

6. (2 marks) The following commands have been entered in MATLAB:

```
A=[5, 6, 9; -10, 2, 7; 1, -5, 15]
c=norm(A,Inf)
```

(A) The value obtained is $c = 31$.
(B) The value obtained is $c = 21$.
(C) The value obtained is $c = 18.8578$.
(D) None of (A)-(C) are true.

ANSWER: **6.** B

7. (2 marks) To plot $y = \cos(x) + \exp(x)x$ for $x \in [-4, 4]$ which of the following commands gives an **error message** in MATLAB:

```
(A) x=linspace(-4,4); f=@(x) cos(x)+exp(x)*x;
    plot(x,f(x))

(B) x=[-4:0.1:4]; y=cos(x)+exp(x).*x;
    plot(x,y)

(C) x=linspace(-4,4); y=@(x) cos(x)+exp(x).*x;
    plot(x, y(x))

(D) None of (A)-(C) are true.
```

ANSWER: **7.** A

8. (2 marks) Entering the commands in MATLAB:

```
p=[1 , -5, 4];
x=roots(p)
```

will give the result

(A) $x = [1; 0.25]$.
(B) $x = [1; 0; -21; 20]$.
(C) $x = [4; 1]$.
(D) None of (A)-(C) are true.

ANSWER: **8.** C

9. (2 marks) Which of the following commands in MATLAB can be used to solve the system $Ax = b$, where A is a matrix and b a vector

```
(A) x=inv(A).*b

(B) x=inverse(A)*b

(C) x=A^{-1}*b

(D) None of (A)-(C) are true.
```

ANSWER: **9.** D

THIS PAGE IS FOR ROUGH WORK

Do not separate the pages