MATH1007 – Notes — By Eric Hua

Contents

Chapter 1 – Functions

1.1 Functions and their graphs

Function: *A function* $y = f(x)$ *from a set* D *to a set* Y *is a rule that assigns a unique element* $f(x) \in Y$ *to each element* $x \in D$. *(*x *is called independent variable,* y *is called dependent variable).*

- *Domain of the function* $y = f(x)$: D = *The set of all values of the independent variable* x *for which the function is defined.*
- *Range of the function:* R = *The set of all values of* $f(x)$ *as* x *varies throughout* D.

Four ways to represent functions:

- *Verbally: by a description in words;*
- *numerically: by a table of values;*
- *visually: by a graph* $\{(x, f(x))|x \in D\}$;
- *algebraically: by an explicit formula.*

Example: $f(x) = \frac{x^2}{x^2-3x+2}$ *is a function,* $D = \{x \neq 1, 2\}$.
Example: $f(x) = \pm x^2$ *is not a function.*
Some special functions:

- *Linear function:* $y = f(x) = mx + b$.
- *Increasing function* $f(x)$: $f(x)$ *increases as* x *increases.*
- *Decreasing function* $f(x)$: $f(x)$ *decreases as* x *increases.*
- *Piecewise defined functions:* $f(x) = \begin{cases} 2x, & x \leq 0; \\ 3x, & x > 0. \end{cases}$
- *Odd functions:* $f(-x) = -f(x)$; *even functions:* $f(-x) = f(x)$.

- *Algebraic function: functions using algebraic operations (+, -, ×, ÷ and taking roots) starting with polynomials. E.g.,*
  - *Power function:* $f(x) = kx^p$, *where* $k \neq 0$ *and* p *are constants.*
    Domain of $f(x) = x^{1/2}$: $\{x \geq 0\}$;
    Domain of $f(x) = x^{-1/2}$: $\{x > 0\}$;
    Domain of $f(x) = x^{-1/3}$: $\{x \neq 0\}$.
  - *Polynomials* $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, *where* n *is a positive integer (which is called the degree of* $P(x)$*).*
  - *Rational function:* $f(x) = \frac{p(x)}{q(x)}$. *Domain* $q(x) \neq 0$.
  - $\sqrt{x-1}$
  - *Absolute value:*

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

- *Transcendental functions (non-algebraic):*
  - *trig functions:* $f(x) = \sin x, \cos x, ...$
  - *exponential functions:* $f(x) = a^x$, $a > 0, a \neq 1$.
  - *logarithmic functions:* $f(x) = \log_a x$, $a > 0, a \neq 1$.

1.2 Combining Functions: Shifting and scaling functions

New Functions:

- *Vertical stretch by a factor of c:* $f(x) \to cf(x)$, $c > 1$;
- *Vertical compress by a factor of* $1/c$: $f(x) \to cf(x)$, $0 < c < 1$;
- *Horizontal stretch by a factor of* $1/c$: $f(x) \to f(cx)$, $0 < c < 1$;
- *Horizontal compress by a factor of c:* $f(x) \to f(cx)$, $c > 1$;
- *Reflection about* x*-axis:* $f(x) \to -f(x)$;
- *Reflection about* y*-axis:* $f(x) \to f(-x)$;
- *Vertical shift up (or down) by* k: $f(x) \to f(x) + k$, $k > 0$ *(or* $k < 0$*)*;
- *Horizontal shift to the right (or left) by* h: $f(x) \to f(x - h)$, $h > 0$ *(or* $h < 0$*)*;
- *Combinations of functions, e.g.,* $f(x)g(x)$;
- *Composite function* $f(g(x))$ *or* $f \circ g(x)$. *The domain of* $f \circ g(x)$ *is the set of all* $x \in D(g)$ *such that* $g(x) \in D(f)$.

Example 1 *Let* $f(x) = \sqrt{2x - 4}$. *Find the new function after shifting downward by 2 units, then shifting left by 3 units, then compressing vertically by 4 units.*

Sol: $\frac{1}{4}(\sqrt{2(x + 3) - 4} - 2)$.

Example 2 *Let* $f(x) = \sqrt{2x - 4}$, $g(x) = \sqrt{3 - 2x}$.

Then $f(g(x)) = \sqrt{2\sqrt{3 - 2x} - 4}$ *with* $D(f \circ g) = \{x \leq -0.5\}$ *(the solution of* $3 - 2x \geq 0$ *and* $2\sqrt{3 - 2x} - 4 \geq 0$*).* $g(f(x)) = \sqrt{3 - 2\sqrt{2x - 4}}$, $D(g \circ f) = \{2 \leq x \leq 3.125\}$ *(the solution of* $2x - 4 \geq 0$ *and* $3 - 2\sqrt{2x - 4} \geq 0$*).*

1.3: Trigonometric functions

Radian $\leftrightarrows$ *Degree:* t *degree* $= \frac{t}{180}\pi$.
Consider a right triangle:

$$\sin t = \frac{\textit{opposite}}{\textit{hypothenuse}}, \quad \cos t = \frac{\textit{adjacent}}{\textit{hypothenuse}}, \quad \tan t = \frac{\sin t}{\cos t},$$

$$\sec t = \frac{1}{\cos t}, \quad \csc t = \frac{1}{\sin t}, \quad \cot t = \frac{1}{\tan t}.$$

Basic relations:

$$\begin{array}{r|l}
\sin(\theta + \frac{\pi}{2}) = +\cos\theta & \sin(\theta+\pi) = -\sin\theta \\
\cos(\theta + \frac{\pi}{2}) = -\sin\theta & \cos(\theta+\pi) = -\cos\theta \\
\tan(\theta + \frac{\pi}{2}) = -\cot\theta & \tan(\theta+\pi) = +\tan\theta \\
\cot(\theta + \frac{\pi}{2}) = -\tan\theta & \cot(\theta+\pi) = +\cot\theta \\
\sec(\theta + \frac{\pi}{2}) = -\csc\theta & \sec(\theta+\pi) = -\sec\theta \\
\csc(\theta + \frac{\pi}{2}) = +\sec\theta & \csc(\theta+\pi) = -\csc\theta
\end{array}$$

Pythagorean trigonometric identity: $\sin^2 x + \cos^2 x = 1$.

Special values:

t	*0*	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	*0*	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	*1*
$\cos t$	*1*	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	*0*

Addition formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

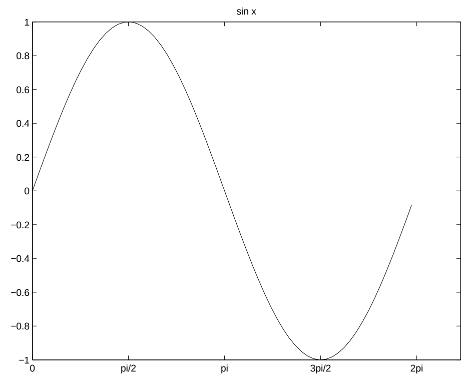
Double-angle formulas:

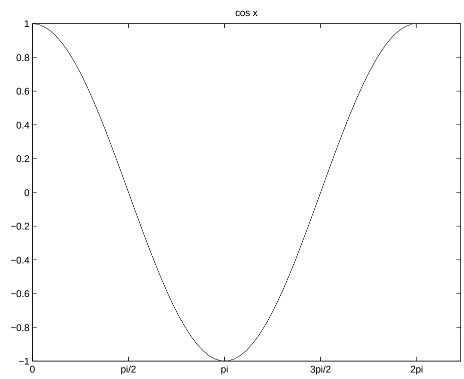
$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x.$$

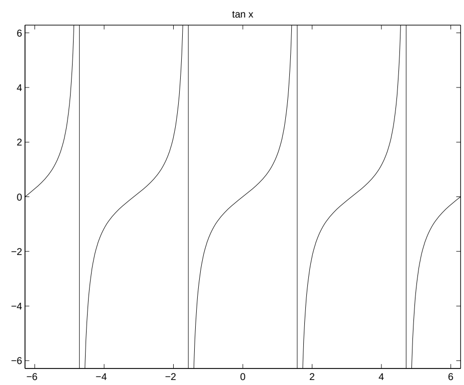
Half-angle formula.

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Periods: $\sin x$ *and* $\cos x$ *have period* 2π, $\tan x$ *and* $\cot x$ *have period* π.
Graphs.







Example 3 *Find all values of* x *in the interval* $[0, 2\pi]$ *such that* $\sin^2 x - 3\cos^2 x = 0$.

Solution: $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$.

Example 4 *Find* $\cos x$ *where* $x \in [\frac{\pi}{2}, 2\pi]$ *such that* $\sin x = 0.8$.

Solution: $\cos x = -0.6$

1.5 Exponential functions

We say that $f(x) = a^x$ *is an exponential function with base* a, $a > 0$, $a \neq 1$. *When* $a = e = 2.71828...$, e^x *is called the natural exponential function.*
Laws of exponents:

$$a^{x+y} = a^x a^y, \quad a^{x-y} = a^x/a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x.$$

General Definition: We say that $P(t)$ *is an exponential function of* t *with base* a *if*

$$P(t) = P_0 a^t, \quad a > 0, a \neq 1,$$

where P_0 *is the initial quantity.*

- *Exponential growth:* $a > 1$;
- *Exponential decay:* $0 < a < 1$.
- $a = P(t+1)/P(t)$.

If we use base $e \doteq 2.71828$, *then*

$$P(t) = P_0 a^t = P_0 e^{kt}, \quad a = e^k,$$

- *Exponential growth:* $k > 0$;
- *Exponential decay:* $k < 0$.
- k *is the continuous growing or decaying rate.*

Special cases:

- *Half-life (exponential decay): The time required for the quantity to be reduced to half. Let* H *be the half-life, then*

$$P(t+H) = \frac{1}{2}P(t) \Rightarrow P(t) = P_0(\frac{1}{2})^{t/H}.$$

- *Doubling-time (exponential growth): The time required for the quantity to be doubled. Let* D *be the doubling time, then*

$$P(t+D) = 2P(t) \Rightarrow P(t) = P_0(2)^{t/D}.$$

Example 5 *A bacterial culture starts with 500 bacteria and doubles in size every hour.*

a) How many are there after t hours?

b) How many are there after 10 minutes?

Solution: a) Let $P(t)$ *be the number after* t *hours. Then* $P(0) = 500$, $P(t+1) = 2P(t)$. $D = 1$.

$$P(t) = (500)2^{t/1} = (500)2^t.$$

b) $P(10/60) = (500)2^{10/60} = (500)2^{1/6}$.

Example 6 *Sketch the graph of* $y = 2^x + 5$.

Example 7 *Sketch the graph of* $y = 2^{-x} + 5$.

1.6 Inverse functions and Logarithms

One-to-one function: $y = f(x)$ *is 1-1* $\Leftrightarrow$ *for each* $y \in R$, *there is only one* $x \in D$. *Horizontal line test can be used to check this.*

Example 8 $f(x) = x^2$ *is not 1-1;* $g(x) = x^2$, $x > 0$ *is 1-1.*

Inverse function: $y = f(x) \to x = f^{-1}(y)$. *We write it as* $y = f^{-1}(x)$.

- *The graph of* f^{-1} *and the graph of* f *are symmetric about the line* $y = x$.
- *Cancelation:* $f(f^{-1}(y)) = y$.
- $f^{-1}(f(x)) = x$
- $D(f) = R(f^{-1})$, $R(f) = D(f^{-1})$.

Example 9 *let* $f(x) = \frac{3x+2}{5x-4}$, *find the inverse* $f^{-1}(x)$.

Strategy:
1) Write $y = \frac{3x+2}{5x-4}$;
2) Switch x *and* y*:* $x = \frac{3y+2}{5y-4}$;
3) Isolate y*:* $y = \frac{4x+2}{5x-3}$;
4) Answer: $y = f^{-1}(x) = \frac{4x+2}{5x-3}$.

$$y = a^x \quad \xrightarrow{inverse function} \quad y = \log_a x,$$

$$y = e^x \quad \xrightarrow{inverse function} \quad y = \log_e x = \ln x,$$

$$y = 10^x \quad \xrightarrow{inverse function} \quad y = \log_{10} x = \log x.$$

Definition: $y = \log_a x$ *is called logarithmic function with the base* a. *Domain* $= \{x > 0\}$.
Properties: Let $B, C > 0$. *Then*

1. $\log_a(BC) = \log_a B + \log_a C$,
2. $\log_a(\frac{B}{C}) = \log_a B - \log_a C$,
3. $\log_a(B^n) = n \log_a B$,
4. $\log_a(a^x) = x$, $\log_a a = 1$,
5. $a^{\log_a B} = B$,
6. $\log_a 1 = 0$.
7. *Change of base:* $\log_a b = \frac{\log_c b}{\log_c a}$.

   Proof. *Let* $x = \log_a b$. *Then* $a^x = b \Rightarrow \log_c a^x = \log_c b \Rightarrow x \log_c a = \log_c b$.

Example 10 *Convert* a^x *to base* e.

$$a^x = e^{x \ln a}.$$

Example 11 *Simplify* $\log_3 18 - \log_3 2$.

Example 12 *Solve for* x:

$$3^{2x-1} = 4, \quad \ln[\ln(2x+1)] = 1, \quad \log_3 x + \log_3(x-8) = 2.$$

Example 13 *Sketch* $y = \ln(x+1) - 2$.

Example 14 *Predict the population in 2010, if*

Year	Population
2000	10
2003	10.5

Sol: Let $P(t)$ be the population after t years. $t = 0 \Leftrightarrow 2000$, $P(0) = P_0 = 10, P(3) = 10.5$.

$$P(t) = P_0 a^t, \Rightarrow P(t) = 10a^t, \Rightarrow P(3) = 10a^3 = 10.5, \Rightarrow a \doteq 1.0164, \Rightarrow P(t) = 10(1.0164)^t.$$

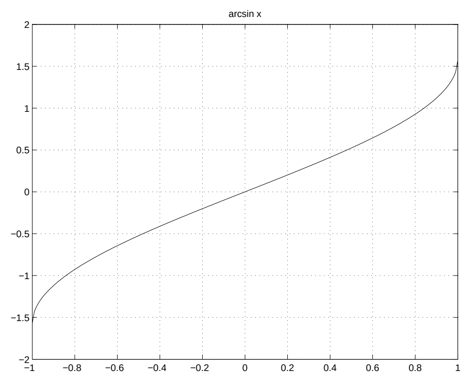
$$P(10) = 10(1.0164)^{10} = 11.76648.$$

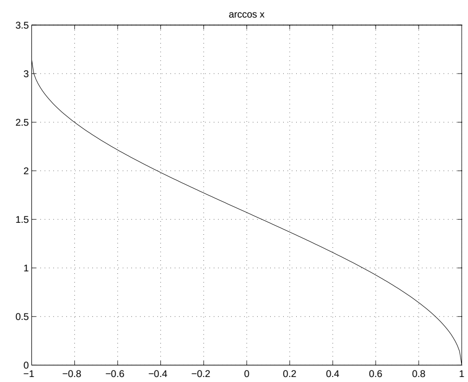
INVERSE TRIG FUNCTIONS:

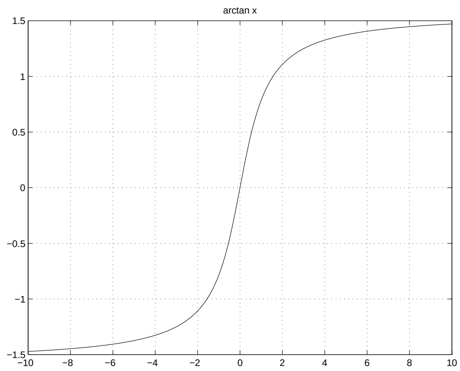
Inverse Trig Function	*Restriction (Range)*	*Meaning*
$t = \arcsin x$ *or* $t = \sin^{-1}(x)$	$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$	$\sin t = x$
$t = \arccos x$ *or* $t = \cos^{-1}(x)$	$0 \le t \le \pi$	$\cos t = x$
$t = \arctan x$ *or* $t = \tan^{-1}(x)$	$-\frac{\pi}{2} < t < \frac{\pi}{2}$	$\tan t = x$

In words: The $t = \arcsin x$ *is an angle (in radians!) whose sin is* x.

Graphs of the inverse functions: *Using the symmetry line* $y = x$.







Example 15 *Find the exact values of the following expressions: (a)* $\arcsin(1)$ *(b)* $\arctan(-1)$ *(c)* $\tan^{-1}(\sqrt{3})$ *(d)*$\sin[\cos^{-1}(\frac{\sqrt{3}}{2})]$

Example 16 *Simplify the following expression:*

$$\tan \arcsin \frac{x}{a}.$$

Solution: Draw a right triangle with hypothenuse a and one side x. Let θ be the opposite angle of x. Then

$$\tan \arcsin \frac{x}{a} = \tan \theta = \frac{x}{\sqrt{a^2 - x^2}}.$$

Example 17 *Find the inverse of* $f(x) = \sin(2e^{3x})$.

Sol: Let $y = \sin(2e^{3x})$*. Switch x and y,*

$$x = \sin(2e^{3y}), \Rightarrow \sin^{-1} x = 2e^{3y}, \Rightarrow \ln(\frac{1}{2} \sin^{-1} x) = 3y, \Rightarrow$$

$$y = \frac{1}{3} \ln(\frac{1}{2} \sin^{-1} x), \Rightarrow f^{-1}(x) = \frac{1}{3} \ln(\frac{1}{2} \sin^{-1} x).$$

Chapter 2 – Limits and Continuity

2.1 Rates of change and tangents to curves

The average rate of change *of* $y = f(x)$ *with respect to* x *over the interval* $[x_1, x_2]$ *is*

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, h \neq 0.$$

Geometrically, it is the slope of the secant through two points $P(x_1, y_1)$ *and* $Q(x_2, y_2)$*.*

Instantaneous rates of change and tangent lines*: What is a tangent line at point* P *on a curve? We chose another point* Q *on the curve. The line* PQ *is called a secant line. When* Q *tends to* P*, the secant* PQ *will tends to a line, which is called a the tangent line of the curve at* P*.*

Example: Estimate the slope of the tangent line to the parabola $y = x^2$ *at the point* $(2, 4)$*.*

Solution:

$$m = \frac{x^2 - 4}{x - 2}.$$

x	m
2.1	4.1
2.01	4.01
2.001	4.001
1.9	3.9
1.99	3.99
1.999	3.999

2.2 The Limit of a Function and limit laws

Definition 1 *We write*

$$\lim_{x \to a} f(x) = L$$

and say "as x approaches a, the limit of $f(x)$ is L." If L is a finite number, we say that the limit exists, otherwise, the limit does not exist.

Example. Estimate the limit of

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

Example. Estimate the limit of

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

x	$\frac{\sin x}{x}$
1	*0.84147098*
0.1	*0.99833417*
0.01	*0.99998333*
0.001	*0.99999983*

Properties: Suppose that $\lim_{x\to a} f(x)$ $\exists$ *and* $\lim_{x\to a} g(x)$ $\exists$.

- $\lim_{x\to a} P(x) = P(a), \quad P(x)$ *is a polynomial.*
- $\lim_{x\to a}(cf(x) \pm dg(x)) = c\lim_{x\to a} f(x) \pm d\lim_{x\to a} g(x), \quad c,\ d$ *are constants.*
- $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad if \ \lim_{x\to a} g(x) \neq 0.$
- $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n.$
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$. *When n is even, we assume that* $\lim_{x\to a} f(x) \neq 0.$

Example 18

$$\lim_{x\to 1}(x^2-3) = 1^2 - 3 = -2, \quad \lim_{x\to 1} \frac{3x^4+8x-2}{x-2} = \frac{3(1)^4+8(1)-2}{1-2} = -9.$$

Special case*:*

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad where\ g(a) = 0.$$

- *If $f(a) \neq 0$, then* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *does not exist.*

- *If $f(a) = 0$, then simplify $\frac{f(x)}{g(x)}$ first, then study the limit.*

Example 19

$$\lim_{x\to 2} \frac{3x^4+8x-2}{x-2} \nexists, \quad \lim_{x\to 2} \frac{x-2}{x-2} = 1.$$

Example 20

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2}(x+2) = 4, \\
\lim_{h\to 0} \frac{(h+1)^2-1}{h} &= \lim_{h\to 0} \frac{h(h+2)}{h} = \lim_{h\to 0}(h+2) = 2, \\
\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} &= \lim_{x\to 0} \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+4}+2)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{4}.
\end{aligned}$$

Theorem 1 *If $f(x) \le g(x)$ near $x = a$, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Theorem 2 *The Sandwich Theorem (The Squeeze Theorem): If $f(x) \le g(x) \le h(x)$ near $x = a$, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Example 21 *Show that*

$$\lim_{x\to 0} x^4 \cos\frac{3}{x} = 0$$

by the Squeeze Theorem.

Hint: $-x^4 \le x^4 \cos\frac{3}{x} \le x^4$.

2.4 One-sided limits

Definition 2 *We write*

$$\lim_{x\to a^-} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the left. Similarly, We write

$$\lim_{x\to a^+} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the right.

Theorem 3

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

Example 22 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t\to 2} H(t) = 1,$$

$$\lim_{t\to 0^+} H(t) = 1, \lim_{t\to 0^-} H(t) = 0, \Rightarrow \lim_{t\to 0} H(t) \nexists.$$

Example 23 $\lim_{x\to 0} \frac{|x|}{x} \nexists.$

$\because \lim_{x\to 0^-} \frac{|x|}{x} = -1, \lim_{x\to 0^+} \frac{|x|}{x} = 1.$

Example 24 *Let*

$$f(x) = \begin{cases} x-5, & x < 0; \\ x^2+3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x\to 0} f(x) \nexists$ *and* $\lim_{x\to 1} f(x) = 4.$

Example 25 *Let*

$$f(x) = \begin{cases} x-5, & x < 0; \\ x^2+3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Calculate $\lim_{x\to 0} f(x)$ *and* $\lim_{x\to 1} f(x)$.

Limits of trig functions*:*

Famous result:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$

This will imply that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} = 0.$$

Example 26

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{3x}{\sin 3x} \cdot \frac{2x}{3x} = \frac{2}{3}.$$

2.5 Continuity

Definition 3 *If* $\lim_{x\to a} f(x) = f(a)$, *then* $f(x)$ *is continuous at* $x = a$, *otherwise,* $f(x)$ *is discontinuous at* $x = a$. *If* $f(x)$ *is continuous at any point on an interval, then* $f(x)$ *is continuous on the interval.*

Example 27 *Explore discontinuity from graph.*

Example 28 *Consider* $f(x) = \frac{x^2-2x+1}{x-1}$ *at* $x = 1$. *Sol:* $f(x)$ *is undefined at* $x = 1$. *But* $\lim_{x\to 1} f(x) = 0$. *So the discontinuous point* $x = 1$ *is* **removable** *if we define* $f(1) = 0$.

Example 29 *Determine the continuity of* $f(x) = \frac{|x|}{x}$.

Sol: $x = 0$ *is not removable.*

Definition 4 *If* $\lim_{x\to a^-} f(x) = f(a)$, *then* $f(x)$ *is continuous from the left at* $x = a$; *if* $\lim_{x\to a^+} f(x) = f(a)$, *then* $f(x)$ *is continuous from the right at* $x = a$.

Example 30 *Determine the left and right continuity at* $x = 0$:

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}.$$

Sol: continuous from right at $x = 0$, *discontinuous from left at* $x = 0$.

Theorem 4 *If* $f(x)$ *and* $g(x)$ *are continuous at* a, *then*

$$f \pm g, fg, cf (c \text{ is a constant}), \frac{f}{g} (\text{if } g(a) \neq 0)$$

are continuous.

Theorem 5 *Polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions and logarithmic functions are continuous in their domain.*

Theorem 6 *If* $\lim_{x\to a} g(x) = b$ *and* $f(x)$ *is continuous at* b, *then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

Furthermore, if $g(x)$ *is continuous at* a, *and* $f(x)$ *is continuous at* $g(a)$, *then* $f(g(x))$ *is continuous at* a.

Example 31

$$\lim_{x\to 1}\arcsin(\frac{1-\sqrt{x}}{1-x}) = \arcsin(\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}) = \arcsin(\frac{1}{2}) = \frac{\pi}{6}.$$

Example 32

$$f(x) = \frac{x+2}{x-4}, g(x) = x^2, \quad \textit{consider } a = 2 \textit{ and } a \neq 2.$$

Theorem 7 *(The Intermediate Theorem) If $f(x)$ is continuous on $[a,b]$, and N between $f(a)$ and $f(b)$, then $\exists c \in [a,b]$ such that $f(c) = N$.*

Example 33

$$\frac{4}{x-5} + \frac{-9x}{(x+1)(x-2)(x+3)} = 0$$

has solutions in $(0,1)$.

Proof. Let $f(x) = \frac{4}{x-5} + \frac{-9x}{(x+1)(x-2)(x+3)}$. Then $f(x)$ is continuous on [0,1]. $f(0) = -4/5$, $f(1) = 1/8$. The conclusion follows from The Intermediate Theorem.

Example 34 *Find k such that $f(x) = \begin{cases} x^3 + kx^2 - 5x, & x > 2; \\ \frac{x}{x-3}, & x \leq 2 \end{cases}$ is continuous at $x = 2$.*

2.6 Limits involving infinity; Asymptotes of graphs

Part 1: Limits at ∞, HA

Definition 5 *The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either*

$$\lim_{x\to\infty} f(x) = L \textit{ or } \lim_{x\to-\infty} f(x) = L.$$

Example 35 *$f(x) = \frac{3x^2-x-1}{2x^2+3x}$ has horizontal asymptote $y = \frac{3}{2}$.*

Example 36 $\lim_{x\to\infty} \frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0} = \begin{cases} 0, & \textit{if } n < m; \\ \frac{a_n}{b_n}, & \textit{if } n = m; \\ \pm\infty, & \textit{if } n > m. \end{cases}$.

Example 37 *$\lim_{x\to\infty}\sin x$, $\lim_{x\to\infty}\cos x$ do not exist.*

Example 38 *$y = \tan^{-1} x$ has horizontal asymptotes $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.*

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad or \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

Example 39 *Find the horizontal asymptotes of the function $f(x) = e^x$.*

Sol: $\lim_{x\to-\infty} e^x = 0$. *Thus, HA:* $y = 0$.

Example 40 *Find the horizontal asymptotes of the function*

$$f(x) = \sqrt{x^2+1} - x.$$

Solution:

$$\lim_{x\to\infty} (\sqrt{x^2+1} - x) = \lim_{x\to\infty} \frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x} = 0.$$

Thus, HA: $y = 0$.

Example 41 *Find the horizontal asymptotes of the function*

$$f(x) = \sqrt{x^2+5x+1} - x.$$

Solution:

$$\lim_{x\to\infty} (\sqrt{x^2+5x+1} - x) = \lim_{x\to\infty} \frac{(\sqrt{x^2+5x+1} - x)(\sqrt{x^2+5x+1} + x)}{\sqrt{x^2+5x+1} + x}$$

$$= \lim_{x\to\infty} \frac{5x+1}{\sqrt{x^2+5x+1} + x} = \frac{5}{2}.$$

Thus, HA: $y = \frac{5}{2}$.

Part 2: Infinite limits, VA

Definition 6

$$\lim_{x\to a} f(x) = \infty$$

means that $f(x)$ can be arbitrarily large as x tends to a;

$$\lim_{x\to a} f(x) = -\infty$$

means that $f(x)$ can be arbitrarily large negative as x tends to a.

Example. $\lim_{x\to 0}\frac{1}{x^2}=\infty$, $\lim_{x\to 1}\frac{-1}{(x-1)^2}=-\infty$.

Definition 7 *The line $x=a$ is called a vertical asymptote of the curve $y=f(x)$ if at least one of the following statements is true:*

$$\lim_{x\to a^-} f(x)=\pm\infty, \lim_{x\to a^+} f(x)=\pm\infty, \lim_{x\to a} f(x)=\pm\infty.$$

Example 42 *Find the infinite limits, limits at infinity, and asymptotes for the function f from its graph.*

Example 43 *$f(x)=\frac{1}{x}$ has HA: $y=0$ and VA: $x=0$.*

Theorem 8 *If $r>0$ is a rational number, then*

$$\lim_{x\to\infty}\frac{1}{x^r}=0.$$

Example 44 *Find the horizontal and vertical asymptotes of the graph of the function*

$$f(x)=\frac{\sqrt{2x^2+1}}{4x-8}.$$

Sol:

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{4x-8}=\frac{\sqrt{2}}{4}, \quad \lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{4x-8}=-\frac{\sqrt{2}}{4}.$$

Thus, HA: $y=\frac{\sqrt{2}}{4}$ and $y=-\frac{\sqrt{2}}{4}$.
VA: $4x-8=0\Rightarrow x=2$.

Example 45 *Evaluate*

$$\lim_{x\to 0^-} e^{1/x}.$$

Sol: Let $t=1/x$, then $x\to 0^-\Leftrightarrow t\to-\infty$.

$$\lim_{x\to 0^-} e^{1/x}=\lim_{t\to-\infty} e^t=0.$$

Part 3: Infinite limits at ∞

The notation $\lim\limits_{x\to\infty} f(x)=\infty$ is used to indicate that the values of f(x) become large as x becomes large. Similar meanings are for

$$\lim_{x\to\infty} f(x)=-\infty, \lim_{x\to-\infty} f(x)=\infty, \lim_{x\to-\infty} f(x)=-\infty.$$

Example 46 $\lim\limits_{x\to\infty} x^5=\infty$, $\lim\limits_{x\to-\infty} x^5=-\infty$.

Example 47 $\lim_{x\to\infty} e^x=\infty$.

Chapter 3. Differentiation Rules

3.1 Tangents and the derivative at a point

Definition 8 *The derivative of the function $y = f(x)$ at a point a is*

$$y'(a) = \frac{dy}{dx} = f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Meaning: $f'(a) =$

- *instantaneous rate of change of $f(x)$ at a, or*
- *rate of change of $f(x)$ at a, or*
- *the slope of the tangent line to the curve at a.*

Example 48 *Let $f(x) = x^2$. Calculate $f'(5)$.*

Sol:

$$f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{(5+h)^2 - 5^2}{h} = 10.$$

Definition 9 *Let $P = (a, f(a))$ be a point on the curve $y = f(x)$. The tangent of $f(x)$ at P is the line through P with slope*

$$m = f'(a) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 49 *Find the slope and the equation of the tangent line to the curve*

$$y = f(x) = 3x^2 - 6x + 1$$

at the point $(2, 1)$. Sketch the curve.

Sol. $a = 2,\ f(a) = 1$.

$$m = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x\to 2} \frac{3x^2 - 6x + 1 - 1}{x-2} = \lim_{x\to 2}(3x) = 6.$$

The tangent line is

$$y - 1 = 6(x-2), \Longrightarrow, y = 6x - 11.$$

To sketch the curve, we complete square: $y = 3(x-1)^2 - 2$.

Example 50 *Find the tangent line to the hyperbola* $xy = 4$ *at the point* $(1, 4)$.

Sol. $a = 1,\ f(x) = y = 1/x$.

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \frac{-1}{1+h} = -1.$$

The tangent line is

$$y = -x + b, \Longrightarrow, 4 = -1 + b, b = 5, \quad y = -x + 5.$$

Example 51 *Find the slope of the tangent line to the curve* $y = \frac{1}{\sqrt{x+1}}$ *at the point* $(0, 1)$.

Sol. $a = 0$,

$$m = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{0+h+1}} - 1}{h} = \lim_{h\to 0} \frac{1 - \sqrt{h+1}}{h\sqrt{h+1}}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{h+1}(1 + \sqrt{h+1})} = -\frac{1}{2}.$$

3.2 The derivative as a function

Definition 10 *The derivative of the function* $y = f(x)$ *is the function* $f'(x)$*:*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Definition 11 *(One sided derivatives):*

- *Right-hand derivative at a:* $f'(a+) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}$.
- *Left-hand derivative at a:* $f'(a-) = \lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h}$.

Example 52 *Let* $f(x) = \sqrt{x-3}$*. Find* $f'(x)$ *and state the domains of* f *and* f'*.*

Sol:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h-3}-\sqrt{x-3}}{h}$$
$$= \lim_{h\to 0}\frac{(\sqrt{x+h-3}-\sqrt{x-3})(\sqrt{x+h-3}+\sqrt{x-3})}{h(\sqrt{x+h-3}+\sqrt{x-3})}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h-3}+\sqrt{x-3})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h-3}+\sqrt{x-3}} = \frac{1}{2\sqrt{x-3}}.$$

The domain of f*:* $x-3\geq 0$*,* $x\geq 3$*.*

The domain of f'*:* $x-3\geq 0$ *and* $2\sqrt{x-3}\neq 0$ *,* $x>3$*.*

Example 53 *Find* $f'(x)$ *from the graph of* f*.*

Definition 12 *The function* f *is differentiable at* a *if* $f'(a)$ *exists. It is differentiable on an interval if* $f'(a)$ *exists for any* a *on the interval.*

Theorem 9 *If a function is differentiable at* $x = c$*, then the function is continuous at* $x = c$*.*

Example 54 $f(x) = |x|$ *is not differentiable at* $x = 0$*.*

Sol:

$$f'(x) = \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0. \end{cases} \qquad f'(0)\nexists.$$

3.3 Differentiation rules

- *Constant rule: If* $f(x) = c$*, then* $f'(x) = 0$ *or* $\frac{d}{dx}(c) = 0$*.*
- *Power Rule: If* $f(x) = x^n$*,* n *is any real number. Then* $f'(x) = nx^{n-1}$*.*
- *Constant multiple rule:* $[cf(x)]' = cf'(x)$*.*
- *Sum rule and difference rule:* $[f(x)\pm g(x)]' = f'(x)\pm g'(x)$
- *Derivative of polynomial:* $[a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0]' = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + ... + a_1$*.*

- *Derivative of exponential function:*

$$(e^x)' = e^x, \quad (a^x)' = a^x \ln a.$$

Example. *Let* $f(x) = 5$. *Find* $f'(x)$.
Example. *Let* $f(x) = x^5$. *Find* $f'(x)$ *and the equation of the tangent at* $(2, 32)$.
Example. *Let* $f(x) = \frac{1}{x^5}$. *Find* $f'(x)$ *and the equation of the tangent at* $(1, 1)$.
Example. *Let* $f(x) = \frac{1}{\sqrt{x}}$. *Find* $f'(x)$ *and the equation of the tangent at* $(1, 1)$.
Example. *Let* $f(x) = \frac{2}{\sqrt{x}}$. *Find* $f'(x)$.

Example 55 *Let* $f(x) = 4x^3 + 6x^2 - 23x + 7$.
1) Find interval(s) such that $f'(x) \leq 1$.
2) Find the equation of the tangent line at $(1, -6)$.

Sol: 1) $f'(x) = 12x^2 + 12x - 23$. *Then from* $f'(x) \leq 1$ *we imply that*

$$12x^2 + 12x - 23 \leq 1, \Rightarrow 12x^2 + 12x - 24 \leq 0, \Rightarrow x^2 + x - 2 \leq 0, \Rightarrow (x-1)(x+2) \leq 0.$$

Therefore, $-2 \leq x \leq 1$.
2) Let $y = mx + b$ *be the tangent line. Then*

$$m = f'(1) = 1, \Rightarrow y = x + b.$$

Sub $(1, -6)$: $-6 = 1 + b, \Rightarrow b = -7, \Rightarrow y = x - 7$.

Example 56 *At what point(s) on the curve* $y = e^x$ *is the tangent line*
a) parallel to $y = 3x - 2$*?*
Solution: $(\ln 3, 3)$.
b) perpendicular to $y = -2x$*?*
Solution: $(-\ln 2, 1/2)$.

The product and quotient rules

- *Product rule:*

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

- *Quotient rule:*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example. Let $f(x) = \sqrt{x}3^x$. Calculate $f'(4)$.
Example. Let $f(x) = (\sqrt{x} + x^2)(x^3 + x)$. Calculate $f'(4)$.
Example. Let $f(x) = (x^3 + 4x^2)(x^5 + x + 1)$. Calculate $f'(1)$ and the tangent at $(1, 15)$.
Example. Let $f(x) = \frac{\sqrt{x}+x^2}{3^x+x}$. Calculate $f'(4)$.
Example. Let $f(x) = \frac{x^3+4x^2}{x^5+x+1}$. Calculate $f'(1)$.

Example 57 *At what point(s) on the curve* $y = \frac{x^2-4}{x+1}$ *is the tangent line*
a) parallel to $y = 3x$?
b) perpendicular to $y = -0.5x$?

Solution: By quotient rule,

$$y' = \frac{(x^2-4)'(x+1) - (x^2-4)(x+1)'}{(x+1)^2} = \frac{2x(x+1) - (x^2-4)1}{(x+1)^2} = \frac{x^2+2x+4}{(x+1)^2}.$$

a) Let $y' = 3 \Rightarrow \frac{x^2+2x+4}{(x+1)^2} = 3 \Rightarrow 2x^2 + 4x - 1 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{6}}{2}$.
b) $(-0.5)y' = -1 \Rightarrow -0.5\frac{x^2+2x+4}{(x+1)^2} = -1 \Rightarrow x^2 + 2x - 2 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{3}}{2}$.

Higher derivatives: *Let* $y = f(x)$. *Then*

$$y''(x) = f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right), \quad y^{(n)}(x) = f^{(n)}(x) = \frac{d}{dx}\left(\frac{dy^{(n-1)}}{dx}\right).$$

Example 58 *Let* $f(x) = 4x^3 + 6x^2 - 23x + 7$. *Then* $f''(x) = 24x + 12$, $f'''(x) = 24$ *and* $f^{(4)}(x) = 0$.

Example 59

$$(x^n)^{(n)} = n!, \quad (\frac{1}{x})^{(n)} = (-1)^n n! x^{-n-1}.$$

Example 60 *Let* $f(x) = \frac{x}{e^x}$. *Calculate* $f^{(n)}(x)$.

Sol:

$$\begin{aligned} f'(x) &= \frac{1-x}{e^x}, \\ f''(x) &= \frac{-(2-x)}{e^x}, \\ f'''(x) &= \frac{(3-x)}{e^x}, \\ &\vdots \\ f^{(n)}(x) &= \frac{(-1)^{n+1}(n-x)}{e^x}. \end{aligned}$$

3.4 The derivative as a rate of change

The derivative $f'(a)$ *is* **the instantaneous rate of change** *of* $f(x)$ *with respect to* x *at* a.

Example 61 *The volume of a sphere of radius* r *is given by*

$$V = \frac{4}{3}\pi r^3.$$

Calculate $\frac{dV}{dr}$ *by definition. What's the meaning of this derivative?*

Sol:

$$\frac{dV}{dr} = \lim_{h\to 0} \frac{V(r+h)-V(r)}{h} = \lim_{h\to 0} \frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h}$$
$$= \frac{4}{3}\pi \lim_{h\to 0} \frac{(r+h)^3 - r^3}{h} = \frac{4}{3}\pi \lim_{h\to 0} \frac{3r^2h + 23rh^2 + h^3}{h} = 4\pi r^2.$$

The derivative is the surface area.

Definition 13 *Let* $s = f(t)$ *be position function.*

$$\textit{average velocity} = \bar{v} = v_{avg} = \frac{\textit{change in distance}}{\textit{change in time}} = \frac{\Delta s}{\Delta t}.$$

Instantaneous velocity, or velocity, or rate of change at t *is*

$$v(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = f'(t).$$

Speed:

$$\textit{speed} = |v(t)|.$$

Acceleration:

$$a(t) = v'(t) = f''(t).$$

Jerk:

$$j(t) = a'(t) = f'''(t).$$

Example 62 *The position of a particle is given by*

$$s = t^3 - 15t^2 + 63t, \quad t \geq 0$$

where s *is measured in meters and* t *in seconds.*

a) What is the initial position? initial velocity? initial acceleration?

b) Find the velocity after 1s and 4s.
c) When is the particle at rest?
d) When is the particle moving in the positive direction?
e) When is the acceleration 0?
f) Find the displacement and the velocity at that time from e).

Solution:

$$s = t^3 - 15t^2 + 63t, \Rightarrow s'(t) = 3t^2 - 30t + 63, \Rightarrow s''(t) = 6t - 30.$$

a) $s(0) = 0, v(0) = s'(0) = 63, a(0) = s''(0) = -30.$
b) $v(1) = s'(1) = 36,\ v(4) = s'(4) = -9.$
c) $s'(t) = 3t^2 - 30t + 63 = 0, \Rightarrow t = 3, 7.$
d) $s'(t) = 3t^2 - 30t + 63 > 0, \Rightarrow 0 < t < 3,\ or\ t > 7.$
e) $s'' = 0 \Rightarrow t = 5.$

Example 63 *Consider the position function*

$$s = t^2 - 3t + 5.$$

Find the velocity at $t = 1$ and $t = 4$, interpret your results.

Sol:

$$v(1) = \lim_{t\to 1} \frac{t^2 - 3t + 5 - 3}{t - 1} = -1.$$

It means move backward.

$$v(4) = \lim_{t\to 1} \frac{t^2 - 3t + 5 - 9}{t - 4} = 5.$$

It means move forward.

Example 64 *A spherical balloon is being inflated. Find the rate of change of the volume with respect to the radius when the radius is 2cm.*

Sol. Let r be the radius, $v(r)$ be the volume. From

$$v(r) = \frac{4}{3}\pi r^3$$

we have

$$\textit{rate of change} = \lim_{\Delta r\to 0} \frac{\Delta v}{\Delta r} = \lim_{r\to 2} \frac{\frac{4}{3}\pi r^3 - \frac{4}{3}\pi 2^3}{r - 2} = \lim_{r\to 2} \frac{\frac{4}{3}\pi(r-2)(r^2 + 2r + 2^2)}{r - 2} = 16\pi.$$

3.5 Derivatives of trigonometric functions

Recall the result:

$$\lim_{h\to 0}\frac{\sin h}{h}=1.$$

Derivative of Trig Functions:

$$(\sin x)'=\cos x,\quad (\cos x)'=-\sin x,\quad (\tan x)'=\sec^2 x,\quad (\sec x)'=\sec x\tan x,...$$

Example 65 *Differentiate* $\csc x$, $\cot x$, $e^x\cos(x)$, $\frac{1+\cos x}{1+\sin x}$, $e^x\sin x$.

Example 66 *Let* $y=\sin(x)$, *calculate* $y^{(10)}(x)$.

Example 67 *Given the position function* $s=f(t)=2\sin(t)$, *calculate the velocity and acceleration at* $t=\frac{\pi}{3}$.

Example 68 *Find the equation of the tangent line to the curve* $\sin(1-x)$ *at* $(1,0)$.

Hint: use the definition of the derivative of the function.

3.6 The chain rule

- *Chain Rule:*

$$[f(g(x))]'=f'(g(x))g'(x),\quad \frac{df(g(x))}{dx}=\frac{df(v)}{dv}\cdot\frac{dg(x)}{dx}, v=g(x),\quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

- *General Power Rule:*

$$[u(x)^n]'=nu^{n-1}u'(x).$$

Example 69 *Let* $f(x)=(x^2-x-1)^{100}$. *Calculate* $f'(x)$.

Sol: $f'(x)=100(x^2-x-1)^{99}(x^2-x-1)'=100(x^2-x-1)^{99}(2x-1)$.

Example 70 *Let* $h(x)=g(f(x))$, *where* $f'(2)=3$, $f(2)=4$, $g'(3)=-5$, $g(4)=8$, $g'(4)=7$. *Find* $h'(2)$.

Solution: $h'(x)=g'(f(x))f'(x)\Rightarrow h'(2)=g'(f(2))f'(2)=g'(4)(3)=7(3)=21$.

Example 71 *Let* $y=\sqrt{x+\sqrt{x^2+x}}$. *Calculate* y'.

Solution:

$$y' = \frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x^2+x}}}(x+\sqrt{x^2+x})'$$

$$= \frac{1}{2\sqrt{x+\sqrt{x^2+x}}}\left(1+\frac{1}{2}\frac{1}{\sqrt{x^2+x}}(x^2+x)'\right) = \frac{1}{2\sqrt{x+\sqrt{x^2+x}}}\left(1+\frac{2x+1}{2\sqrt{x^2+x}}\right)$$

Example 72 *Find $f'(x)$. If*

$$f(x) = \sin x^2, \quad \sin^2 x, \quad e^{\sin x}, \quad \sin(\cos(\tan x))).$$

3.7 Implicit differentiation

Implicit Differentiation: Assume $f(x,y) = C$. To find y',

- *consider x as an independent variable, y as a dependent variable;*
- *differentiate both sides with respect to x;*
- *isolate y'.*

Example 73 *Find y' from $y^2 + x^2 = 1$.*

Sol:

$$\frac{d}{dx}(y^2+x^2) = \frac{d1}{dx}, \Rightarrow 2yy' + 2x = 0, \Rightarrow y' = -\frac{x}{y}.$$

Example 74 *Let*

$$y^2 + x^2 = xy + 3.$$

1) Find the equation of the tangent line to the curve at $(0, \sqrt{3})$.

2) Find all the points on the curve where the tangent line is either horizontal or vertical.

Sol: 1)

$$\frac{d}{dx}(y^2+x^2) = \frac{d}{dx}(xy+3), \Rightarrow 2yy' + 2x = y + xy', \Rightarrow y' = \frac{y-2x}{2y-x}.$$

$$m = y'|_{(0,\sqrt{3})} = 0.5, \Rightarrow y = 0.5x + \sqrt{3}.$$

2) Horizontal tangent line: $y' = 0 \Rightarrow y - 2x = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1, y = 2$ *or* $x = -1, y = -2$.

Vertical tangent line: $y' = \infty \Rightarrow 2y - x = 0 \Rightarrow y^2 = 1 \Rightarrow y = 1, x = 2$ *or* $y = -1, x = -2$.

Example 75 *Find* y' *from* $\tan(xy + x) = x + y$.

Normal line: *The normal line of* $f(x)$ *at* $(a, f(a))$ *is the line which is perpendicular to the tangent line at the point.*

Example 76 *Let* $y^2 + x^2 = xy + 3$. *Find the equation of the normal line to the curve at* $(0, \sqrt{3})$.

3.8 Derivative of logarithmic function

A general formula for the derivative of inverse functions:

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}.$$

Proof. *Let* $y = f^{-1}(x)$, *then* $f(y) = x \Rightarrow f'(y)y' = 1$.

Some special results:

- *Derivatives of log functions:*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad (\ln f(x))' = \frac{f'(x)}{f(x)},$$

$$(\log_a |x|)' = \frac{1}{x \ln a}, \quad (\log_a f(x))' = \frac{f'(x)}{f(x) \ln a}, \ldots$$

Change base:

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

Example 77 *Differentiate* $\ln(x^2 + 1)$.

Logarithmic differentiation

Example 78 *Differentiate* $y = \frac{x^2+x+5}{(x+1)^2}$, x^x, $(\sin x)^x$.

Number e

$$e = \lim_{x \to 0}(1 + x)^{1/x} = \lim_{x \to \infty}(1 + \frac{1}{x})^x.$$

3.9 Derivative of inverse trigonometric functions

Derivative of inverse trig functions:

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d \arccos x}{dx} = \frac{1}{-\sqrt{1-x^2}}, \quad \frac{d \arctan x}{dx} = \frac{1}{1+x^2}.$$

Example 79 $y = \sin(\arctan 2x)$, $y = \arcsin\left(\frac{b+a\cos x}{a+b\cos x}\right)$.

3.11 Linearization and differentials

LINEAR APPROXIMATIONS: we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

Definition 14 *The approximation*

$$f(x) \approx f(a) + f'(a)(x-a)$$

is called the linear approximation or tangent line approximation of f at a.

$$L(x) = f(a) + f'(a)(x-a)$$

is called the linearization of f at a.

Example 80 *a) Find the linearization of the function $f(x) = \sqrt{x}$ at $a = 9$ and use it to approximate the numbers $\sqrt{9.01}$.*

Sol:

$$\sqrt{x} \approx 3 + \frac{1}{6}(x-9) = \frac{x}{6} + \frac{3}{2}, \qquad \sqrt{9.01} \approx 3 + \frac{0.01}{6}.$$

b) Are these approximations overestimates or underestimates?

Sol: over.

c) For what values of x is the linear approximation in a) accurate to within 0.1?

Example 81 *The linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$.*

Definition 15 *If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable. That is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation $dy = f'(x)dx$.*

Remark. *dy represents the amount that the tangent line rises or falls (the change in the linearization).* Δy *represents the amount that the curve* $y = f(x)$ *rises or falls when changes by an amount dx.*

Example 82 *Compare the values of* Δy *and* dy *if* $y = f(x) = x^3 + x^2 - 2x + 1$ *and* x *changes from: 2 to 2.01.*

Sol. We have:

$$f(2) = 2^3 + 2^2 - 2(2) + 1 = 9,$$

$$f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1 = 9.140701,$$

$$\Delta y = f(2.01) - f(2) = 0.140701,$$

In general,

$$dy = f'(x)dx = (3x^2 + 2x - 2)dx.$$

When $dx = \Delta x = 0.01$,

$$dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14.$$

Remark. In the notation of differentials, the linear approximation can be written as:

$$f(a + dx) \approx f(a) + dy.$$

Example 83 *The radius of a sphere was measured to be 21 cm with a possible error of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?*

Sol. This can be approximated by the differential

$$dV = 4\pi r^2 dr.$$

When r = 21 and dr = 0.05, this becomes:

$$dV = 4\pi(21)^2 0.05 \approx 277.$$

Chapter 4. Applications of Derivatives

4.1 Extreme values of functions

- *Absolute (Global) Maximum and Minimum:* $f(x)$ *has a Global (Absolute) Maximum at* p *if* $f(p) \geq f(x)$ *for all* x *in the domain;* $f(x)$ *has a Global (Absolute) Minimum at* p *if* $f(p) \leq f(x)$ *for all* x *in the domain;*
- *Local (or relative) extreme:* $f(x)$ *has a local minimum at* p *if* $f(p) \leq f(x)$ *for points* x *near* p*;* $f(x)$ *has a local maximum at* p *if* $f(p) \geq f(x)$ *for points* x *near* p*;*
- *Critical point (critical number): A point* p *in the domain such that* $f'(p) = 0$ *or* $f'(p)$ *undefined is called a critical number,* $(p, f(p))$ *is a critical point,* $f(p)$ *is a critical value.*

EXTREME VALUE THEOREM: If f(x) is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a, b].
FERMAT's THEOREM: If f *has a local maximum or minimum at* c*, and if* $f'(c)$ *exists, then* $f'(c) = 0$*.*
CLOSED INTERVAL METHOD: To find a global maximum or minimum for $f(x)$ *on a closed interval* $[a, b]$*:*

1. *Find all the critical numbers, e.g.,* x_1*, ...,* x_n*.*
2. *global minimum* $= \min\{f(x_1), ..., f(x_n), f(a), f(b)\}$*;*
   global maximum $= \max\{f(x_1), ..., f(x_n), f(a), f(b)\}$*.*

Example 84 *Find the critical numbers of* $f(x) = x^{3/5}(4 - x)$*.*

Sol: the critical numbers are 3/2 and 0.

Example 85 *Find the global maximum and minimum of the function*

$$f(x) = 2x^3 - 3x^2 - 12x + 7, \quad [-2, 0].$$

Sol: Step 1) $f'(x) = 6x^2 - 6x - 12$, $f'(x) = 0 \Rightarrow x = -1, 2$, $f'(x)$ *is defined anywhere. Hence* $x = -1$ *is the only one critical number in* $(-2, 0)$.

Step 2) global minimum $= \min\{f(-2), f(-1), f(0)\} = \min\{3, 14, 7\} = 3$;

global maximum $= \max\{f(-2), f(-1), f(0)\} = \max\{3, 14, 7\} = 14$.

Example 86 *Find the global maximum and minimum of the function*

$$f(x) = x^2 e^{2x}, \quad [-2, 2].$$

Example 87 *An object has the position function*

$$s(t) = e^{-t} \cos t.$$

Find the maximum distance to $s(0)$ *for* $t \geq 0$.

Sol: Let

$$d(t) = |s(t) - s(0)| = |e^{-t} \cos t - 1| = 1 - e^{-t} \cos t.$$

We just need to find the global maximum to $d(t)$. $d'(t) = e^{-t}(\cos t + \sin t)$.

$$d'(t) = 0 \Rightarrow \cos t + \sin t = 0, \Rightarrow t = k\pi - \frac{\pi}{4}, k = 1, 2, 3, ...$$

global maximum $= \max\{d(0), d(k\pi - \pi/4), k = 1, 2, 3, ...\} = d(3\pi/4)$.

4.2 The Mean Value Theorem

ROLLE's THEOREM: Let f be a function that satisfies the following three hypotheses:

1. *f is continuous on the closed interval [a, b]*

2. *f is differentiable on the open interval (a, b)*

3. f(a) = f(b)

Then, there is a number c in (a, b) such that $f'(c) = 0$.

Remark. There is at least one point (c, f(c)) on the graph where the tangent is horizontal. *MEAN VALUE THEOREM: Let f be a function that satisfies the following two hypotheses:*

1. *f is continuous on the closed interval [a, b]*
2. *f is differentiable on the open interval (a, b)*

Then, there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example. Suppose that f(0) = -3 and $f'(x) \leq 5$ *for all values of x. How large can f(2) possibly be?*

Sol:

$$f(2) = f(0) + f'(c)(2 - 0) = -3 + 2f'(c) \leq -3 + 10 = 7.$$

Theorem. If $f'(x) = 0$ *for all x in an interval (a, b), then f is constant on (a, b); If* $f'(x) = g'(x)$ *for all x in an interval (a, b), then* $f(x) - g(x)$ *is constant on (a, b).*

Example. Simplify

$$f(x) = tan^{-1}x + cot^{-1}x.$$

Sol. Since $f'(x) = 0$, $f(x) = c$, $c = f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. $\therefore f(x) = \frac{\pi}{2}$.

4.3 Monotonic functions and the first derivative test

INCREASING/DECREASING TEST (I/D TEST): If $f'(x) > 0$ *on an interval, then f is increasing on that interval. If* $f'(x) < 0$ *on an interval, then f is decreasing on that interval.* **First Derivative Test***: Let p be a critical number. If* f' *changes from - to + at p, then f has a local minimum at p; If* f' *changes from + to - at p, then f has a local maximum at p.* ***Example 88*** *Let* $f(x) = x^3 - 3x^2$.

(a) Find all the critical numbers.

(b) State all the intervals of increase and decrease.

(c) Find all the local minimum points and all the local maximum points.

Solution:

(a) $f'(x) = 3x^2 - 6x = 3x(x-2)$. *Let* $f'(x) = 0$. *We have* $3x(x-2) = 0$, *which gives* $x = 0, 2$.

(b) Look at the following table

x	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
$f'(x)$	+	-	+
$f(x)$	*increase*	*decrease*	*increase*

Therefore,

The intervals of increase: $-\infty < x < 0$, $2 < x < \infty$.

The intervals of decrease: $0 < x < 2$

(c) Note that at $x = 0$, $f'(x)$ *changes from + to -; at* $x = 2$, $f'(x)$ *changes from - to +. By the First Derivative Test,* $f(x)$ *has a local maximum at* $x = 0$ *and a local minimum at* $x = 2$.

Example 89 *Let* $g(x) = x + 2\sin x$, $0 \le x \le 2\pi$.

(a) Find all the critical numbers.

(b) State all the intervals of increase and decrease.

(c) Find all the local minimum points and all the local maximum points.

Solution:

(a) $g'(x) = 1 + 2\cos x$, $g'(x) = 0 \Rightarrow x = 2\pi/3, 4\pi/3$.

(b) Look at the following table

x	$0 < x < 2\pi/3$	$2\pi/3 < x < 4\pi/3$	$4\pi/3 < x < 2\pi$
$f'(x)$	+	-	+
$f(x)$	*increase*	*decrease*	*increase*

Therefore,

The intervals of increase: $0 < x < 2\pi/3$, $4\pi/3 < x < 2\pi$.

The intervals of decrease: $2\pi/3 < x < 4\pi/3$

(c) Note that at $x = 2\pi/3$, $f'(x)$ *changes from + to -; at* $x = 4\pi/3$, $f'(x)$ *changes from - to +. By the First Derivative Test,* $f(x)$ *has a local maximum at* $x = 2\pi/3$ *and a local minimum at* $x = 4\pi/3$.

4.4 Concavity and curve sketching

Definition 16 *(CONCAVITY)If the graph of f lies above all of its tangents on an interval I, it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.*

CONCAVITY TEST: If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 17 *Point of inflection: If $f(x)$ changes concavity at p, then p is an inflection point, and $f''(p) = 0$ or undefined.*

Second Derivative Test*: Let p be a critical number. If $f''(p) > 0$, then f has a local minimum at p; If $f''(p) < 0$, then f has a local maximum at p; If $f''(p) = 0$, then nothing.*

Example. Let $f(x) = x^3 - 3x^2$.

(c) Find all the local minimum points and all the local maximum points by Second Derivative Test.

(d) Find all the points of inflection.

(e) State intervals of concavity.

(f) Sketch the graph.

Solution:

(c) From $f'(x) = 3x^2 - 6x$ we get $f''(x) = 6x - 6$. $f'(x) = 0 \Rightarrow x = 0, 2$. Note that $f''(0) = -6 < 0$ $f''(2) = 6 > 0$ By the Second Derivative Test, $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$.

(d) $f''(x) = 0 \Rightarrow x = 1$.

x	$-\infty < x < 1$	$1 < x < \infty$
$f''(x)$	-	+
$f(x)$	*concave down*	*concave up*

Since $f(x)$ changes concavity at $x = 1$, $x = 1$ is a point of inflection.

(e) Concave up: $1 < x < \infty$; Concave down: $-\infty < x < 1$.

Example 90 *Consider the function*

$$f(x) = \frac{x}{x^2 - 1}.$$

Study the concavity and find all the points of inflection.

Solution: The domain of the function: $x \neq \pm 1$.

$$f'(x) = \frac{-1 - x^2}{(x^2-1)^2}, \quad f'' = \frac{2x(x^2+3)}{(x^2-1)^3}.$$

$$f''(x) = 0, \Rightarrow x = 0.$$

x	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f''(x)$	-	+	-	+
$f(x)$	*concave down*	*concave up*	*concave down*	*concave up*

Example 91 *Sketch $f(x)$, which satisfies:*

1. $D = \{x \neq 0\}$.
2. $\lim_{x \to \pm\infty} f(x) = 1, \quad \lim_{x\to 0^+} f(x) = -\infty, \quad \lim_{x \to 0^-} f(x) = \infty.$
3. $f'(2) = f'(-2) = 0$; $f'(x) > 0$ *when* $0 < x < 2$ *and* $-2 < x < 0$; $f'(x) < 0$ *when* $x > 2$ *and* $x < -2$.
4. $f''(-3) = f''(3) = 0$; $f''(x) > 0$ *when* $x > 3$ *and* $-3 < x < 0$; $f''(x) < 0$ *when* $x < -3$ *and* $0 < x < 3$.
5. $f(2) = 2, f(-2) = -1$.

Example 92 *Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.*

Sol: Notice that the domain of f is $\{x | x \neq 0\}$. So, we check for vertical asymptotes by computing the left and right limits as $x \to 0$.

$$\lim_{x\to 0^+} e^{1/x} = \infty, \quad \lim_{x \to 0^-} e^{1/x} = 0.$$

This shows that $x = 0$ is a vertical asymptote.

$$\lim_{x \to \pm\infty} e^{1/x} = 1,$$

this shows that $y = 1$ is a horizontal asymptote.

The Chain Rule gives:

$$f'(x) = -\frac{e^{1/x}}{x^2},$$

we have $f'(x) < 0$ for all $x \neq 0$. Thus, f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number. So, the function has no maximum or minimum.

$$f''(x) = \frac{e^{1/x}(2x+1)}{x^4},$$

$f''(x) > 0$ when $x > -1/2 (x \neq 0)$, and $f''(x) < 0$ when $x < -1/2$. So, the curve is concave downward on $(-\infty, -1/2)$ and concave upward on $(-1/2, 0)$ and on $(0, \infty)$.

The inflection point is $(-1/2, e^{-2})$.

Example 93 *Show that $e^x \geq 1 + x$ for $x \geq 0$.*

Proof. *Let $f(x) = e^x - (1 + x)$. Then $f'(x) = e^x - 1 \geq 0$ when $x \geq 0$. Thus $f(x)$ is increasing when $x \geq 0$. Note that $f(0) = 0$, so $f(x) \geq 0$ for $x \geq 0$.*

The following checklist is intended as a guide to sketching a curve y = f(x) by hand.

Not every item is relevant to every function. For instance, a given curve might not have an asymptote or possess symmetry. However, the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

- *A. DOMAIN*
- *B. INTERCEPTS*
- *C. SYMMETRY*
    1. *EVEN FUNCTION: f(-x) = f(x) for all x in D. the curve is symmetric about the y-axis. This means that our work is cut in half.*
    2. *ODD FUNCTION: f(-x) = -f(x) for all x in D. the curve is symmetric about the origin. This means that our work is cut in half.*
    3. *PERIODIC FUNCTION: f(x + p) = f(x) for all x in D, where p is a positive constant. The smallest such number p is called the period.*
- *D. ASYMPTOTES*
    - *HORIZONTAL:* $\lim_{x\to\pm\infty} f(x) = L$, *then* $y = L$ *is a HA.*
    - *VERTICAL:* $\lim_{x\to a^\pm} f(x) = \pm\infty$, *then* $x = a$ *is a VA.*
- *E. INTERVALS OF INCREASE OR DECREASE: use I/D Test.*
- *F. LOCAL MAXIMUM AND MINIMUM VALUES: First Derivative Test or Second Derivative Test.*

- *G. CONCAVITY AND POINTS OF INFLECTION*

Example 94 *Consider the function*

$$f(x) = \left[x^2(6-x)\right]^{1/3}, \qquad -\infty < x < \infty.$$

We have (you don't need to check the following results!)

$$f'(x) = \frac{4-x}{\sqrt[3]{x(6-x)^2}}, \quad f''(x) = \frac{-8}{\sqrt[3]{x^4(6-x)^5}}.$$

a) Find all the critical numbers of $f(x)$.

b) Find all the intervals of increasing and decreasing, and classify all the critical numbers as local maxima, minima, or neither.

c) Study the concavity and find the point(s) of inflection.

d) Sketch the graph. (Remark. $f(x)$ has no vertical and horizontal asymptotes).

Sol: a) From $f'(x) = 0$ we get $x = 4$;
From $f'(x)$ undefined we imply that $x = 0, 6$.
So we have three critical numbers $x = 4, 0, 6$.

b) Look at the following table

x	$-\infty < x < 0$	$0 < x < 4$	$4 < x < 6$	$6 < x < \infty$
$f'(x)$	-	+	-	-
$f(x)$	*decreasing*	*increasing*	*decreasing*	*decreasing*

By the First Derivative Test, $f(0) = 0$ is a local minimum, $f(4) = \sqrt[3]{32}$ is a local maximum, $f(6)$ is neither.

c) Note that $f''(x)$ undefined at $x = 0, 6$. Look at the following table

x	$-\infty < x < 0$	$0 < x < 6$	$6 < x < \infty$
$f''(x)$	-	-	+
$f(x)$	*concave down*	*concave down*	*concave up*

$x = 6$ or $(6, f(6))$ is a point of inflection.

Example 95 *Sketch the graph of:*

$$f(x) = \frac{2x^2}{x^2-4}, \quad f(x) = xe^x, \quad f(x) = \frac{\sin x}{2+\cos x}, \quad f(x) = \ln(1-x^2).$$

Example 96 *Sketch the graph of f if f satisfies all of the following:*

(i) $f'(x) > 0$ *on* $(-\infty, 1)$, $f'(x) < 0$ *on* $(1, \infty)$

(ii) $f''(x) > 0$ *on* $(-\infty, -2)$ *and* $(2, \infty)$, $f''(x) < 0$ *on* $(-2, 2)$

(iii) $\lim_{x\to-\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 0$.

4.5 Indeterminate Forms and L'Hospital's Rule

In this section, we are going to deal with the limit with the form:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^{\infty}, \quad 0\cdot\infty, \quad 0^0, \ldots$$

L'Hospital's rule: If $\frac{f(x)}{g(x)}$ *becomes* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *as* $x \to x_0$, *where* x_0 *is finite or* ∞, *then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Remark. $x \to x_0$ *can be replaced by any of the symbols* $x \to x_0^+$, $x \to x_0^-$, $x \to \infty$, *or* $x \to -\infty$.

Example 97 *Calculate*

$$\lim_{x\to 0} \frac{\sin x}{x}, \quad \lim_{x\to 0} \frac{\sin x}{x^2}, \quad \lim_{t\to 0} \frac{e^t - t - 1}{t^2}, \quad \lim_{x\to\infty} \frac{(\ln x)^2}{x^2}.$$

Example 98 *Calculate*

$$\lim_{x\to\infty} x^2 e^{-x}, \quad \lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^x.$$

Solution:

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \lim_{x\to\infty} \frac{0}{e^x} = 0.$$

To solve the second limit, let $y = \left(1 - \frac{1}{x}\right)^x$, *then*

$$\ln y = x \ln\left(1 - \frac{1}{x}\right) = \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}}.$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{-\frac{1}{x^2}(1-\frac{1}{x})} = -1. \Rightarrow$$

$$\lim_{x\to\infty} y = \frac{1}{e}.$$

4.8 Antiderivatives

Definition 18 *A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.*

Some basic results:

function	*antiderivative*	*formula*
k	$kx + C$	$\int kdx = kx + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C;\ (n \neq -1)$
e^{kx}	$\frac{1}{k}e^{kx} + C$	$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$
a^{kx}	$\frac{a^{kx}}{k\ln a} + C$	$\int a^{kx}dx = \frac{a^{kx}}{k\ln a} + C$
$\frac{1}{x}$	$\ln\|x\| + C$	$\int \frac{1}{x}dx = \ln\|x\| + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$	$\int \cos kxdx = \frac{1}{k}\sin kx + C$
$\sin kx$	$-\frac{1}{k}\cos kx + C$	$\int \sin kxdx = -\frac{1}{k}\cos kx + C$
$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	$\int \sec^2 kxdx = \frac{1}{k}\tan kx + C$
$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	$\int \sec kx \tan kxdx = \frac{1}{k}\sec kx + C$
$\frac{1}{\sqrt{1-(kx)^2}}$	$\frac{1}{k}\arcsin kx + C$	$\int \frac{1}{\sqrt{1-(kx)^2}}dx = \frac{1}{k}\arcsin kx + C$
$\frac{1}{1+(kx)^2}$	$\frac{1}{k}\arctan kx + C$	$\int \frac{1}{1+(kx)^2}dx = \frac{1}{k}\arctan kx + C$
		$\int kf(x)dx = k\int f(x)dx$
		$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx.$

Example 99 $\int \frac{x^2-1}{x^3}dx = \int (x^{-1} - x^{-3})dx = \ln|x| + \frac{1}{2x^2} + C.$

$\int \sin 4x + e^{5x}dx = -\frac{1}{4}\cos 4x + \frac{1}{5}e^{5x} + C.$

Example 100 *Find $f(x)$ such that*

$$f'(x) = \sin x + \frac{4x^2 - 22}{x^3}.$$

Sol:

$$f'(x) = \sin x + 4x^{-1} - 22x^{-3}, \Rightarrow f(x) = -\cos x + 4\ln|x| + 11x^{-2} + C.$$

RECTILINEAR MOTION: Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. The position function is an antiderivative of the velocity function. The velocity function is an antiderivative of the acceleration.

Example 101 *An object moves along a coordinate line with velocity*

$$v(t) = 2 - 3t + 3t^2 \quad \textit{units/s.}$$

Its initial position is 2 units to the right of the origin(when t=0). Find the position of the object and acceleration after 4s.

Sol: Let $s(t)$ *be the position. Then* $s(0) = 2$. *Since* $s'(t) = v(t)$,

$$s(t) = 2t - \frac{3}{2}t^2 + t^3 + C.$$

$s(0) = 2 \Rightarrow C = 2$ *and*

$$s(t) = 2t - \frac{3}{2}t^2 + t^3 + 2. \Rightarrow$$
$$s(4) = 50, a(4) = v'(4) = 21.$$

Chapter 5. Integrals

5.1 Area and estimating with finite sums

We need three formulas for calculating the sum:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Three ways to estimate the area of the region S bounded by the continuous function $y = f(x)$ (where $f(x) \geq 0$), $x = a$, $x = b$ and the x-axis:

We divide the interval $[a, b]$ into n equal parts with endpoints $x_0 = a$, $x_1 = a + \frac{b-a}{n}$, $x_2 = a + \frac{2(b-a)}{n}$,..., $x_n = a + \frac{n(b-a)}{n} = b$, $\Delta x = \frac{b-a}{n}$,

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = [f(x_0) + f(x_1) + ... + f(x_{n-1})]\,\Delta x,$$
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = [f(x_1) + ... + f(x_{n-1}) + f(x_n)]\,\Delta x,$$
$$M_n = \sum_{i=1}^{n} f(\frac{x_{i-1}+x_i}{2})\Delta x = \left[f(\frac{x_0+x_1}{2}) + f(\frac{x_1+x_2}{2}) + ... + f(\frac{x_{n-1}+x_n}{2})\right]\Delta x.$$

Here L_n is called Left-hand Sum, R_n is Right-hand Sum, M_n is called Midpoint Sum, or Midpoint Rule.
Example 102 *Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.*

Sol: a) Suppose we divide S into 4 strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = 1/4$, $x = 2/4$, and $x = 3/4$.

a.1) If we approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip, then

$$R_4 = \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{2}{4})^2 + \frac{1}{4}(\frac{3}{4})^2 + \frac{1}{4}(1)^2 = \frac{15}{32}.$$

a.2) If we approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the left edge of the strip, then

$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{2}{4})^2 + \frac{1}{4}(\frac{3}{4})^2 = \frac{7}{32}.$$

b) Suppose we divide S into n strips S_1, ..., S_n by drawing the vertical lines $x = 1/n$, $x = 2/n$, ..., $x = n/n$.

b.1) If we approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip, then

$$R_n = \frac{1}{n}(\frac{1}{n})^2 + \frac{1}{n}(\frac{2}{n})^2 + ... + \frac{1}{n}(\frac{n}{n})^2 = \frac{1}{n^3}(1^2 + 2^2 + ... + n^2)$$

$$= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \to \frac{1}{3}.$$

b.2) If we approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the left edge of the strip, then

$$L_n = \frac{1}{n}(\frac{0}{n})^2 + \frac{1}{n}(\frac{1}{n})^2 + ... + \frac{1}{n}(\frac{n-1}{n})^2 = \frac{1}{n^3}(1^2 + 2^2 + ... + (n-1)^2)$$

$$= \frac{1}{n^3}\frac{(n-1)(n)(2n-1)}{6} \to \frac{1}{3}.$$

b.3) If we approximate each strip by a rectangle whose base is the same as the strip and whose height is the y value at the midpoint x of the strip, then

$$M_n = \frac{1}{n}(\frac{1}{2n})^2 + \frac{1}{n}(\frac{3}{2n})^2 + ... + \frac{1}{n}(\frac{2n-1}{2n})^2 = \frac{1}{4n^3}(1^2 + 3^2 + ... + (2n-1)^2)$$

$$= \frac{1}{4n^3}\left[\frac{(2n)(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{6}\right] \to \frac{1}{3}.$$

Now, let's consider the distance problem:

Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

If the velocity remains constant, then

Distance=(velocity)(time).

5.2 Sigma notation and limits of finite sum

$$GRS(\textit{General Riemann Sum}) = \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \leq c_i \leq x_i, \Delta x = \frac{b-a}{n}.$$

Definition 19 *The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is:*

$$Area = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n.$$

For given velocity function $y = v(t)$, total distance from $t = a$ to $t = b$ is the area under the curve $y = v(t)$.

$$d = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n,$$

where we replace x by t and f by v in the definition.

5.3 The Definite Integral

Definition 20 *Definite integral*

$$\int_a^b f(x)dx = F(b) - F(a), \quad F'(x) = f(x).$$

Definite integral = limit of Riemann sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty} GRS,$$

where

$$GRS(\textit{General Riemann Sum}) = \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \leq c_i \leq x_i, \Delta x = \frac{b-a}{n}.$$

The relation to area is:

$$\int_a^b f(x)dx = \textit{area above } x\textit{-axis - area below } x\textit{-axis}.$$

Example 103 *Calculate* $\int_0^5 f(x)dx$, *where*

$$f(x) = \begin{cases} x, & 0 \le x \le 1; \\ 1, & 1 \le x \le 2; \\ 3 - x, & 2 \le x \le 3; \\ -\sqrt{1-(x-4)^2}, & 3 \le x \le 5. \end{cases}$$

Solution:

$$\int_0^5 f(x)dx = \text{area above } x\text{-axis - area below } x\text{-axis}$$
$$= \frac{(1+3)1}{2} - \frac{1}{2}\pi(1)^2 = 2 - \frac{\pi}{2}.$$

Some basic properties about definite integral:

- $\int_a^b c dx = c(b-a)$;
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$;
- $\int_a^a f(x)dx = 0$;
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$;
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;
- *Constant multiple:* $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;
- *Comparison of Definite Integrals: If* $f(x) \le g(x)$ *for* $a \le x \le b$, *then*

  $$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

  In particular, if $m \le f(x) \le M$, *then*

  $$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Example 104 *Let* $\int_1^5 f(x)\,dx = 3$, $\int_1^5 g(x)\,dx = 5$. *Calculate* $\int_1^5 [2f(x) - g(x) - 1]\,dx$.

Answer:

$$\int_1^5 [2f(x) - g(x) - 1]\,dx = 2\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx - \int_1^5 1\,dx$$
$$= 2(3) - 5 - 1(5-1) = -3.$$

Example 105 *Find an upper bound and a lower bound to*

$$\int_{\pi/3}^{5\pi/6} \sin x \, dx.$$

Solution: Let $f(x) = \sin x$. $f'(x) = \cos x = 0 \Rightarrow x = \frac{\pi}{2}$. *By Closed Interval Method,*

$$\max f(x) = \max\{f(\frac{\pi}{2}), f(\frac{\pi}{3}), f(\frac{5\pi}{6})\} = \max\{1, \frac{\sqrt{3}}{2}), \frac{1}{2}\} = 1,$$

$$min\, f(x) = min\,\{f(\frac{\pi}{2}), f(\frac{\pi}{3}), f(\frac{5\pi}{6})\} = min\{1, \frac{\sqrt{3}}{2}), \frac{1}{2}\} = \frac{1}{2}.$$

Thus

$$\frac{1}{2}(\frac{5\pi}{6} - \frac{\pi}{3}) \leq \int_{\pi/3}^{5\pi/6} \sin x \, dx \leq 1(\frac{5\pi}{6} - \frac{\pi}{3}), i.e,$$

$$\frac{\pi}{4} \leq \int_{\pi/3}^{5\pi/6} \sin x \, dx \leq \frac{\pi}{2}.$$

If $f(x) \geq 0$ *and integrable over* $[a, b]$*, then the area under the curve* $y = f(x)$ *over* $[a, b]$ *is:*

$$Area = \int_a^b f(x)dx.$$

Example 106 *Find the area under the curve* $f(x) = 1/x$ *over* $[1, 2]$.

Average value of $f(x)$ **from** a **to** b =

$$\frac{1}{b-a} \int_a^b f(x)dx.$$

Example 107 . *Find the average value of* x^3 *over* $[0, 2]$.

5.4 The Fundamental Theorem of Calculus

- *FTC1: The (first) Fundamental Theorem of Calculus: If*

  $$g(x) = \int_a^x f(t)dt,$$

  then $g'(x) = f(x)$;

- *FTC2: If $F'(x) = f(x)$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 108

$$\begin{aligned}
\frac{d}{dx}\int_0^x f(t)dt &= f(x), \\
\frac{d}{dx}\int_0^{x^2} f(t)dt &= \frac{d}{du}\int_0^u f(t)dt \cdot \frac{du}{dx} = 2xf(x^2), \quad u = x^2, \\
\frac{d}{dx}\int_{x^2}^{x^3} f(t)dt &= \frac{d}{dx}\left(\int_{x^2}^0 f(t)dt + \int_0^{x^3} f(t)dt\right) = -2xf(x^2) + 3x^2 f(x^3).
\end{aligned}$$

Example 109 *Let $g(x) = \int_a^x e^{t^2} dt$. Calculate $g'(2)$ and $g''(2)$.*

Example 110 *Calculate $\int_0^2 3^t dt$.*

Solution: Let $f(t) = 3^t$, then $F(t) = \frac{1}{\ln 3}3^t + C$.

$$\int_0^2 3^t dt = F(2) - F(0) = \frac{8}{\ln 3}.$$

Example 111 *Let $f'(x) = x^3$, $f(0) = 1$. Calculate $f(2)$.*

Solution:

$$f(b) = f(a) + \int_a^b f'(x)dx \Rightarrow f(2) = f(0) + \int_0^2 f'(x)dx = 1 + \frac{1}{4}x^4|_0^2 = 5.$$

Example 112 *An object moves along a coordinate line with velocity*

$$v(t) = 2 - 3t + t^2 \quad \text{units/s}.$$

Its initial position is 2 units to the right of the origin(when t=0). Find the position of the object and total distance it traveled 4s later.

Sol: Let $s(t)$ be the position. Then $s(0) = 2$. Since $s'(t) = v(t)$,

$$s(t) = \int s'(t)dt = \int v(t)dt = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^4 + C.$$

$s(0) = 2 \Rightarrow C = 2$ *and*

$$s(t) = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^4 + 2. \Rightarrow s(4) = 7\frac{1}{3}.$$

Note that

$$\int_a^b |v(t)|dt = \textit{distance traveled from } t = a \textit{ to } t = b,$$

$$\int_0^4 |v(t)|dt = \int_0^4 |2 - 3t + t^2|dt.$$

$$|2 - 3t + t^2| = |(t-1)(t-2)| = \begin{cases} 2 - 3t + t^2, & 0 < t < 1; \\ -(2 - 3t + t^2), & 1 < t < 2; \\ 2 - 3t + t^2, & 2 < t < 4. \end{cases}$$

$$\int_0^4 |v(t)|dt = \int_0^4 |2 - 3t + t^2|dt$$

$$= \int_0^1 (2 - 3t + t^2)dt + \int_1^2 -(2 - 3t + t^2)dt + \int_2^4 (2 - 3t + t^2)dt.$$

Example 113 *Find the area of the region between the x-axis and the graph of* $f(x) = x^3 - 2x^2 - 3x$, $-1 \le x \le 3$.

5.6 Area Between Curves

Theorem. If $f(x) \ge g(x)$ *for* $a \le x \le b$*, then the area of the region bounded by*

$$y = f(x), y = g(x), x = a, x = b$$

is

$$\int_a^b (f(x) - g(x))dx.$$

Example 114 *Calculate the area of the region bounded by*

$$y = x^2 - 4x + 7, y = -x^2 + 4x + 1, x = 0, x = 2.$$

Sol: Step 1. Find intersections: Let $(-x^2 + 4x + 1) = (x^2 - 4x + 7), \Rightarrow x^2 - 4x + 3 = 0, \Rightarrow x = 1, x = 3$.

Step 2. By using the intersections, the interval $(0, 2)$ *is divided into* $(0, 1)$ *and* $(1, 2)$*. In* $(0, 1)$*:* $x^2 - 4x + 7 > -x^2 + 4x + 1$*; In* $(1, 2)$*:* $-x^2 + 4x + 1 > x^2 - 4x + 7$*. Therefore*

$$area = \int_0^1 [(x^2 - 4x + 7) - (-x^2 + 4x + 1)]dx + \int_1^2 [(-x^2 + 4x + 1) - (x^2 - 4x + 7)]dx$$

$$= \int_0^1 (2x^2 - 8x + 6)dx + \int_1^2 (-2x^2 + 8x - 6)dx.$$

Example 115 *Find the area of the region between* $y = x^{1/2}$ *and* $y = x^{1/3}$ *for* $0 \le x \le 1$.

Example. Find the area of the region bounded by the parabolas $y = 2x - x^2$ *and* $y = x^2$.

Sol: Step 1. Find intersections: $2x - x^2 = x^2 \Rightarrow x = 0, x = 1$.

Step 2. For $0 < x < 1$, $2x - x^2 > x^2$.

$$area = \int_0^1 [2x - x^2 - x^2]dx = 1 - \frac{2}{3} = \frac{1}{3}.$$

Example 116 *Calculate the area of the region bounded by*

$$y = \sin x, y = \cos x, x = 0, x = \frac{\pi}{2}.$$

Sol: $A = 2\sqrt{2} - 2$.

Example 117 *Find the area of the region bounded by*

$$y = e^x, y = 4e^{-x}, y = 1.$$

Example 118 *Calculate the area of the region bounded by*

$$y = \sin x, y = \cos x, \quad \text{between } x = 0 \text{ and } x = \pi/2.$$

Theorem. If the region is bounded by

$$x = f(y), x = g(y), y = c, y = d$$

where $f(y) \ge g(y)$ *for* $c \le y \le d$, *then the area of the region is*

$$A = \int_c^d (f(y) - g(y))dy.$$

Example . Calculate the area of the region enclosed by

$$y = x - 1, y^2 = 2x + 6.$$

Sol:

$$A = \int_{-2}^4 [(y - 1) - (\frac{1}{2}y^2 - 3)]dy = 18.$$