MCG 4102 / 5108 - Assignment 3 Solutions

1) With $[N(\xi,\eta)] = [\frac{1}{4}(1-\xi)(1-\eta) \quad \frac{1}{4}(1+\xi)(1-\eta) \quad \frac{1}{4}(1+\xi)(1+\eta) \quad \frac{1}{4}(1-\xi)(1+\eta)]$,
one easily gets
$[\frac{\partial N(\xi,\eta)}{\partial \xi}] = [-\frac{1}{4}(1-\eta) \quad \frac{1}{4}(1-\eta) \quad \frac{1}{4}(1+\eta) \quad -\frac{1}{4}(1+\eta)]$ and
$[\frac{\partial N(\xi,\eta)}{\partial \eta}] = [-\frac{1}{4}(1-\xi) \quad -\frac{1}{4}(1+\xi) \quad \frac{1}{4}(1+\xi) \quad \frac{1}{4}(1-\xi)]$.
We have $x(\xi,\eta)$ and $y(\xi,\eta)$, and inversely, $\xi(x,y)$ and $\eta(x,y)$.
Using the chain rule of differentiation for each of the entries $N_m(\xi,\eta)$ of $[N(\xi,\eta)]$, with $m = 1,2,3,4$,
$\frac{\partial N_m(\xi,\eta)}{\partial \xi} = \frac{\partial N_m(\xi,\eta)}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_m(\xi,\eta)}{\partial y}\frac{\partial y}{\partial \xi}$. Similarly, $\frac{\partial N_m(\xi,\eta)}{\partial \eta} = \frac{\partial N_m(\xi,\eta)}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_m(\xi,\eta)}{\partial y}\frac{\partial y}{\partial \eta}$.
In matrix form,

$$\begin{Bmatrix} \frac{\partial N_m}{\partial \xi} \\ \frac{\partial N_m}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_m}{\partial x} \\ \frac{\partial N_m}{\partial y} \end{Bmatrix},$$

and since $x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)x_i$ and $y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)y_i$,

$$[J(\xi,\eta)] = \begin{bmatrix} [\frac{\partial N(\xi,\eta)}{\partial \xi}]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} & [\frac{\partial N(\xi,\eta)}{\partial \xi}]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \\ [\frac{\partial N(\xi,\eta)}{\partial \eta}]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} & [\frac{\partial N(\xi,\eta)}{\partial \eta}]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

2) a) With $[J] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det[J] = ad - bc$, and $[J] = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$[B(\xi,\eta)] = [J]^{-1}\begin{bmatrix} [\frac{\partial N(\xi,\eta)}{\partial \xi}] \\ [\frac{\partial N(\xi,\eta)}{\partial \eta}] \end{bmatrix}$$

$$[B(\xi,\eta)] = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} [-\frac{1}{4}(1-\eta) & \frac{1}{4}(1-\eta) & \frac{1}{4}(1+\eta) & -\frac{1}{4}(1+\eta)] \\ [-\frac{1}{4}(1-\xi) & -\frac{1}{4}(1+\xi) & \frac{1}{4}(1+\xi) & \frac{1}{4}(1-\xi)] \end{bmatrix}.$$

2) b) Apply the above with $\xi_1 = -0.57735, \eta_1 = -0.57735$, and $w_1^{\xi} = w_1^{\eta} = 1.00000$.

2) c) $$[B(\xi_1,\eta_1)]^T[B(\xi_1,\eta_1)]\det[J(\xi_1,\eta_1)] = \begin{bmatrix} 0.2608 & -0.0403 & -0.0699 & -0.1506 \\ & 0.1553 & 0.0108 & -0.1258 \\ & & 0.0187 & 0.0403 \\ \text{Sym} & & & 0.2359 \end{bmatrix}.$$

3) $[K^e] = \sum_{j=1}^{M}\sum_{i=1}^{N} w_i^{\xi} w_j^{\eta}[B(\xi_i,\eta_j)]^T[B(\xi_i,\eta_j)]\det[J(\xi_i,\eta_j)]$ means

$$
\begin{aligned}
[K^e] = \quad & w_1^{\xi} w_1^{\eta} [B(\xi_1, \eta_1)]^T [B(\xi_1, \eta_1)] \det[J(\xi_1, \eta_1)] \;. \\
+ \quad & w_2^{\xi} w_1^{\eta} [B(\xi_2, \eta_1)]^T [B(\xi_2, \eta_1)] \det[J(\xi_2, \eta_1)] \\
+ \quad & w_1^{\xi} w_2^{\eta} [B(\xi_1, \eta_2)]^T [B(\xi_1, \eta_2)] \det[J(\xi_1, \eta_2)] \\
+ \quad & w_2^{\xi} w_2^{\eta} [B(\xi_2, \eta_2)]^T [B(\xi_2, \eta_2)] \det[J(\xi_2, \eta_2)]
\end{aligned}
$$

Finally, $[K^e] = \begin{bmatrix} 0.5798 & 0.0337 & -0.3404 & -0.2730 \\ & 0.7772 & -0.4326 & -0.3782 \\ & & 0.8191 & -0.0461 \\ \text{Sym} & & & 0.6974 \end{bmatrix}.$