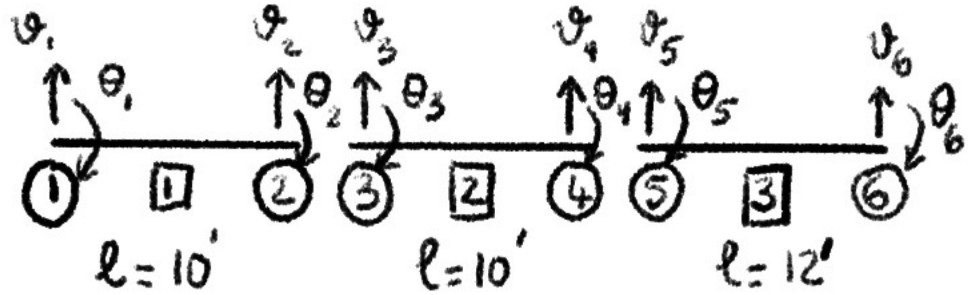
MCG 4102 / 5108 - Assignment 2 Solutions

1) For now, the 3 elements are defined independently (they "do not know each other").



For a beam element, $[K^e] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Sym} & & & 4L^2 \end{bmatrix}$.

For elements 1 and 2, $L = 10'$, therefore, $[K^{(1)}] = [K^{(2)}] = \frac{EI}{1{,}000}\begin{bmatrix} 12 & 60 & -12 & 60 \\ & 400 & -60 & 200 \\ & & 12 & -60 \\ \text{Sym} & & & 400 \end{bmatrix}$.

For element 3, $L = 12'$. To convert $\frac{EI}{1{,}728}[\bullet]$ to $\frac{EI}{1{,}000}[\bullet]$, divide all entries by $1{,}728$. Then,

$$[K^{(3)}] = \frac{EI}{1{,}000}\begin{bmatrix} 6.9 & 41.6 & -6.94 & 41.6 \\ & 333.3 & -41.6 & 166.6 \\ & & 6.9 & -41.6 \\ \text{Sym} & & & 333.3 \end{bmatrix}.$$

2) To build the assemblage stiffness matrix $[K^a]$ for the whole beam, we need to enforce that $v_2 = v_3 = v_B$, $\theta_2 = \theta_3 = \theta_B$, $v_4 = v_5 = v_C$ and $\theta_4 = \theta_5 = \theta_C$. This is done by overlapping the matrices, and adding corresponding terms. Note that this also holds for force and moments. Now, the elements "know each other"!

$$[K^a] = \frac{EI}{1{,}000}\begin{bmatrix} 12 & 60 & -12 & 60 & 0 & 0 & 0 & 0 \\ & 400 & -60 & 200 & 0 & 0 & 0 & 0 \\ & & 24 & 0 & -12 & 60 & 0 & 0 \\ & & & 800 & -60 & 200 & 0 & 0 \\ & & & & 18.9 & -18.3 & -6.94 & 41.6 \\ & & & & & 733 & -41.6 & 166.6 \\ & & & & & & 6.9 & -41.6 \\ \text{Sym} & & & & & & & 333.3 \end{bmatrix}, \text{ such that}$$

$$[K^a]\begin{Bmatrix} v_A \\ \theta_A \\ v_B \\ \theta_B \\ v_C \\ \theta_C \\ v_D \\ \theta_D \end{Bmatrix} = \begin{Bmatrix} F_A \\ M_A \\ F_B \\ M_B \\ F_C \\ M_C \\ F_D \\ M_D \end{Bmatrix}.$$

3) At $A,$ no deflection, no rotation from fixation, i.e. $v_A = \theta_A = 0$. At $C,$ no deflection due to roller, i.e. $v_B = 0.$
Therefore, the rows and colums corresponding to v_A, θ_A and v_B can be removed to produce a solvable system of equations $[K]\{U\} = \{F\}$.

$$[K] = \frac{EI}{1{,}000}\begin{bmatrix} 24 & 0 & 60 & 0 & 0 \\ & 800 & 200 & 0 & 0 \\ & & 733.3 & -41.6 & 166.6 \\ & & & 6.9 & -41.6 \\ \text{Sym} & & & & 333.3 \end{bmatrix}.$$

Since $M_B = -500$ ft.lb and $F_D = -2{,}000$ lbf are the only external loads, $\{F\} = \begin{Bmatrix} 0 \\ -500 \\ 0 \\ -2{,}000 \\ 0 \end{Bmatrix}$.

Finally, $\begin{Bmatrix} v_B \\ \theta_B \\ \theta_C \\ v_D \\ \theta_D \end{Bmatrix} = [K]^{-1}\begin{Bmatrix} 0 \\ -500 \\ 0 \\ -2{,}000 \\ 0 \end{Bmatrix} = \frac{1{,}000}{EI}\begin{Bmatrix} 298.4 \text{ ft} \\ 29.2 \text{ rad} \\ -119.4 \text{ rad} \\ -2{,}585 \text{ ft} \\ -263.4 \text{ rad} \end{Bmatrix}.$

4) Going back to the individual elements in the assemblage, let's consider element 1. This element is in equilibrium such that $[K^{(1)}]\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$, with $v_1 = v_A = 0,\ \theta_1 = \theta_A = 0,$ and $v_2 = v_B,\ \theta_2 = \theta_B$.
From known $v_A, \theta_A, v_B, \theta_B$ and $[K^{(1)}]$, we get $F_1 = -1{,}827$ lbf, $M_1 = -12{,}060$ ft.lb, $F_2 = 1{,}827$ lbf, $M_2 = -6{,}210$ ft.lb.

5) We are interested in the reaction forces F_A, F_C and reaction moment M_A. To obtain them, let's extract the corresponding lines of the assemblage system in Question 2.

$$\frac{EI}{1{,}000}\begin{bmatrix} 12 & 60 & -12 & 60 & 0 & 0 & 0 & 0 \\ 60 & 400 & -60 & 200 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -60 & -18.9 & -18.3 & -6.9 & 41.6 \end{bmatrix}] \begin{Bmatrix} v_A \\ \theta_A \\ v_B \\ \theta_B \\ v_C \\ \theta_C \\ v_D \\ \theta_D \end{Bmatrix} = \begin{Bmatrix} F_A \\ M_A \\ F_C \end{Bmatrix}.$$

This yields $F_A = -1{,}828$ lbf, $M_A = -12{,}062$ ft.lb, and $F_C = 3{,}828$ lbf.

6) Assuming that the cross section A of the beam is rectangular of height $2c,$ the maximum bending stress at location i is $\sigma = \frac{M_i c}{I},$ and the maximum shear stress is $\tau = \frac{3F_i}{2A}.$ Check your strength of materials or machine design course notes for more details.