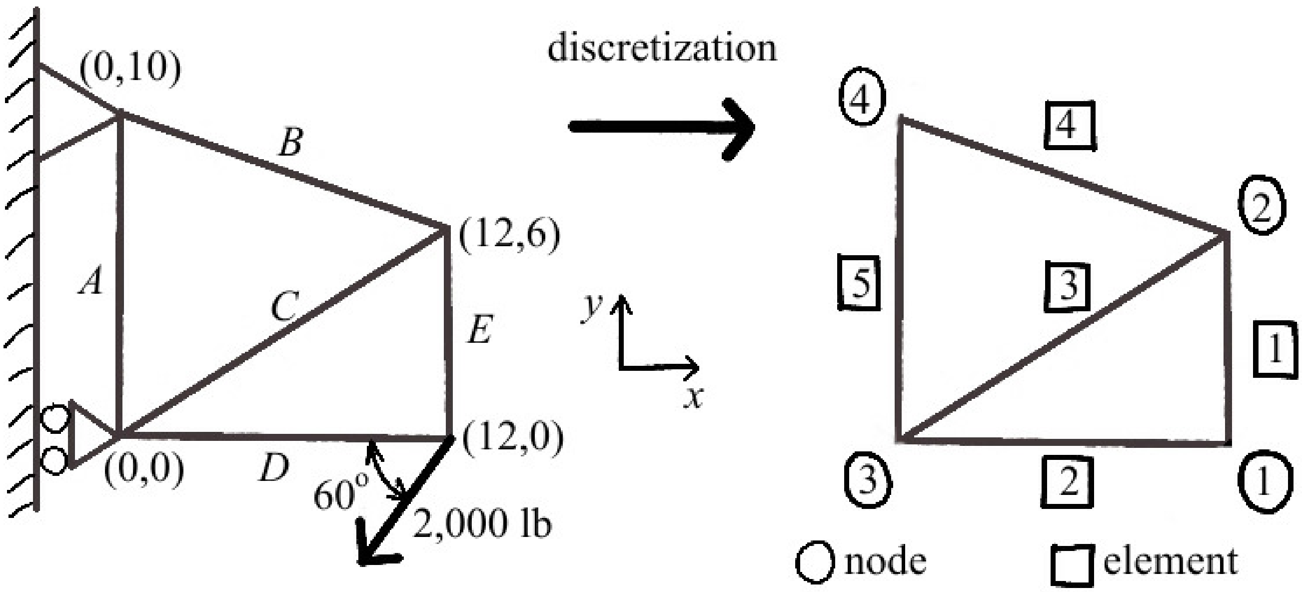
MCG 4102 / 5108 - Assignment 1 Solutions

Using the truss example given in class (4 nodes, and 5 elements involving 2 materials):



1) For element 1 in its local coordinate system, $\left[K^{(1)'}\right] = \frac{A^{(1)}E^{(1)}}{L^{(1)}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Material flag set to 1: 0.50-in steel: $A^{(1)} = \frac{\pi}{4}(0.5)^2 = 0.196\,\text{in}^2$

$E^{(1)} = 30 \times 10^6$ psi

$L^{(1)} = \sqrt{(12-12)^2 + (6-0)^2} = 6\,\text{in}$

so that $\frac{A^{(1)}E^{(1)}}{L^{(1)}} = 10^3 \times 982$ lbf/in.

In the global coordiante system, $[K^{(1)}] = \frac{A^{(1)}E^{(1)}}{L^{(1)}}\begin{bmatrix} n_{11}^2 & n_{21}n_{11} & -n_{11}^2 & -n_{21}n_{11} \\ n_{11}n_{21} & n_{21}^2 & -n_{11}n_{21} & -n_{21}^2 \\ -n_{11}^2 & -n_{21}n_{11} & n_{11}^2 & n_{21}n_{11} \\ -n_{11}n_{21} & -n_{21}^2 & n_{11}n_{21} & n_{21}^2 \end{bmatrix}$, with

$n_{11} = \frac{x_j - x_i}{L^{(1)}} = \frac{12-12}{6} = 0$. Finally, $[K^{(1)}] = 10^3\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 982 & 0 & -982 \\ 0 & 0 & 0 & 0 \\ 0 & -982 & 0 & 982 \end{bmatrix}$ lbf/in.

$n_{21} = \frac{y_j - y_i}{L^{(1)}} = \frac{6-0}{6} = 1$

2) Let us first create a null assemblage stiffness matrix $[K^a]$ involving global node numbers 1 to 4, with 2 degrees of freedom per node.

$$
\begin{array}{c}
\qquad\quad \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
[K^a] = \begin{bmatrix}
0_{2x2} & 0_{2x2} & 0_{2x2} & 0_{2x2} \\
0_{2x2} & 0_{2x2} & 0_{2x2} & 0_{2x2} \\
0_{2x2} & 0_{2x2} & 0_{2x2} & 0_{2x2} \\
0_{2x2} & 0_{2x2} & 0_{2x2} & 0_{2x2}
\end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

At node 1, stiffness contribution from elements 1 and 2.
At node 2, stiffness contribution from elements 1, 3 and 4.
At node 3, stiffness contribution from elements 2, 3 and 5.
At node 4, stiffness contribution from elements 4 and 5. Then,

$$
\begin{array}{c}
\qquad\quad \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
[K^a] = \begin{bmatrix}
K^{(1)}_{1,1} + K^{(2)}_{1,1} & K^{(1)}_{1,2} & K^{(2)}_{1,3} & 0_{2x2} \\
K^{(1)}_{2,1} & K^{(1)}_{2,2} + K^{(3)}_{2,2} + K^{(4)}_{2,2} & K^{(3)}_{2,3} & K^{(4)}_{2,4} \\
K^{(2)}_{3,1} & K^{(3)}_{3,2} & K^{(2)}_{3,3} + K^{(3)}_{3,3} + K^{(5)}_{3,3} & K^{(5)}_{3,4} \\
0_{2x2} & K^{(4)}_{4,2} & K^{(5)}_{4,3} & K^{(4)}_{4,4} + K^{(5)}_{4,4}
\end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

Given $[K^{(1)}] = 10^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 982 & 0 & -982 \\ 0 & 0 & 0 & 0 \\ 0 & -982 & 0 & 982 \end{bmatrix}$ lbf/in, $[K^{(2)}] = 10^3 \begin{bmatrix} 115 & 0 & -115 & 0 \\ 0 & 0 & 0 & 0 \\ -115 & 0 & 115 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ lbf/in,

$[K^{(3)}] = 10^3 \begin{bmatrix} 351 & 176 & -351 & -176 \\ 176 & 88 & -176 & -88 \\ -351 & -176 & 351 & 176 \\ -176 & -88 & 176 & 88 \end{bmatrix}$ lbf/in, $[K^{(4)}] = 10^3 \begin{bmatrix} 98 & -33 & -98 & 33 \\ -33 & 11 & 33 & -11 \\ -98 & 33 & 98 & -33 \\ 33 & -11 & -33 & 11 \end{bmatrix}$ lbf/in

and $[K^{(5)}] = 10^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 589 & 0 & -589 \\ 0 & 0 & 0 & 0 \\ 0 & -589 & 0 & 589 \end{bmatrix}$ lbf/in, we obtain

$$
[K^a] = 10^3 \begin{bmatrix}
115 & 0 & 0 & 0 & -115 & 0 & 0 & 0 \\
 & 982 & 0 & -982 & 0 & 0 & 0 & 0 \\
 & & 450 & 143 & -351 & -176 & -98 & 33 \\
 & & & 1080 & -176 & -88 & 33 & -11 \\
 & & & & 466 & 176 & 0 & 0 \\
 & & & & & 677 & 0 & -589 \\
 & & & & & & 98 & -33 \\
\text{Sym} & & & & & & & 600
\end{bmatrix} \text{lbf/in}
$$

3) Becasue of boundary conditions $u_3 = u_4 = v_4 = 0,$ the rows and columns corresponding to these degrees of freedom can be removed from $[K^a].$ We are left with

$$[K]\{U\} = \{F\} \text{ or } 10^3 \begin{bmatrix} 115 & 0 & 0 & 0 & 0 \\ & 982 & 0 & -982 & 0 \\ & & 450 & 143 & -176 \\ & & & 1080 & -88 \\ \text{Sym} & & & & 677 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -1,000 \\ -1,732 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

4) $\{U\} = [K]^{-1}\{F\}$, which yields

$$u_1 = -0.00879\,\text{in} \quad u_2 = 0.01012\,\text{in}$$
$$v_1 = -0.03581\,\text{in} \quad v_2 = -0.03405\,\text{in}$$
$$u_3 = 0 \quad u_4 = 0$$
$$v_3 = -0.00180\,\text{in} \quad v_4 = 0$$

5) In element 3, by definition, the axial extension $\delta = u'_j - u'_i = u'_2 - u'_3$ with $u'_2 = n_{11}u_2 + n_{21}v_2$ and $u'_3 = n_{11}u_3 + n_{21}v_3$. For element 3, $n_{11} = \frac{x_2 - x_3}{L^{(3)}} = \frac{12-0}{13.42} = 0.8944$, and $n_{21} = \frac{y_2 - y_3}{L^{(3)}} = \frac{6-0}{13.42} = 0.4472$.
Then, $\delta = -0.0057$ in. Strain $\varepsilon = \frac{\delta}{L^{(3)}}$, with $L^{(3)} = \sqrt{(0-12)^2 + (0-6)^2} = 13.42$ in, from which $\varepsilon = -0.4 \times 10^{-3}$ in/in. Stress $\sigma = E\varepsilon$, with $E^{(3)} = 30 \times 10^6$ psi, from which $\sigma = -12,004$ psi (compression).
Finally, $F = \sigma A^{(3)}$, with $A^{(3)} = \frac{\pi}{4}(0.5)^2 = 0.196\,\text{in}^2$, from which $F^{(3)} = -2352\,\text{lbf}$ (compressive force).

Note: Numerical values may change slightly due to round-off errors.