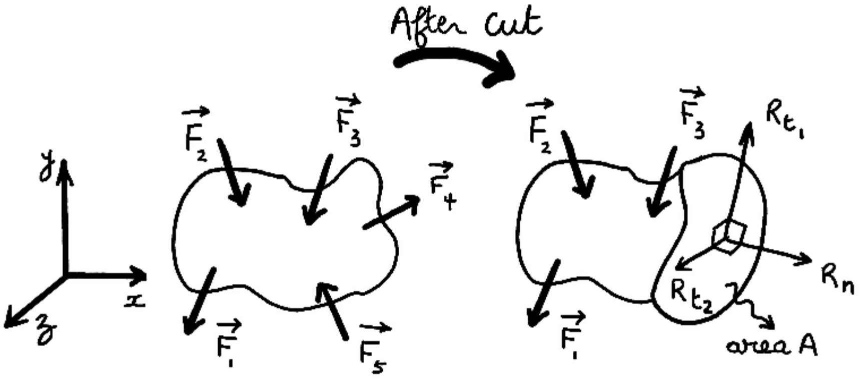
Linear elasticity - Principle of virtual displacements

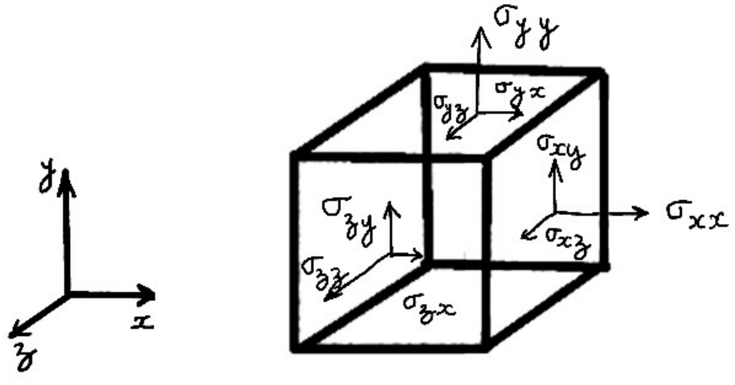
This is a short review of linear elasticity in statics for use in finite element formulation.

1) Stress at a point

Consider a body in static equilibrium, loaded as shown below:



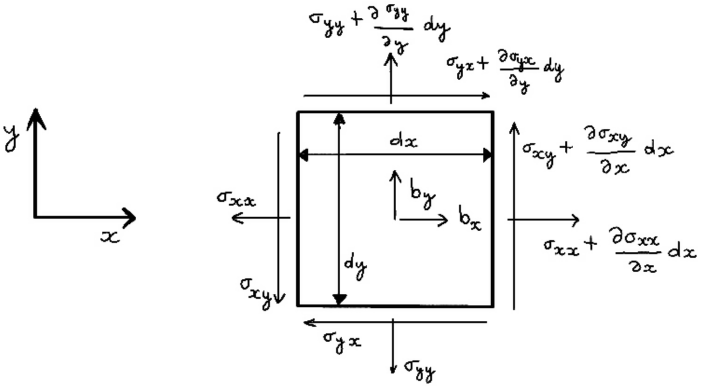
The external forces $\vec{F_1}$, $\vec{F_2}$, ... are transmitted through the deformable body in a complex manner. When the body is cut along some plane, a force $\vec{R}$ is required to maintain static equilibrium. $\vec{R}$ has a normal component R_n and two tangential orthogonal components R_{t1} and R_{t2}. Consider small area ΔA instead of A. Then, ΔR_n, ΔR_{t1}, ΔR_{t2} act on ΔA. The normal stress σ_n is defined as $\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta R_n}{\Delta A}$ and the two shear (tangential) stresses as $\sigma_{t1} = \lim_{\Delta A \to 0} \frac{\Delta R_{t1}}{\Delta A}$ and $\sigma_{t2} = \lim_{\Delta A \to 0} \frac{\Delta R_{t2}}{\Delta A}$. For an infinitesimal volume element of a body positioned at a point in a global (x, y, z) coordinate system:



(1^{st} subscript: facet normal; 2^{nd} subscript: direction)

2) Equations of static equilibrium

Consider an infinitesimal 2-D volume element of thickness t. It is assumed that the normal and shear stresses vary from point to point in the body in some continuous manner. Therefore, a first order Taylor series expansion is used.



b_x, b_y : components of body force per unit volume.

Equilibrium in x-direction:

$$(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx)dy - \sigma_{xx}dy + (\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy)dx - \sigma_{yx}dx + b_x dxdy = 0$$

Equilibrium in y-direction:

$$(\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy)dx - \sigma_{yy}dx + (\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x} dx)dy - \sigma_{xy}dy + b_y dxdy = 0$$

Finally, in 2-D, $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + b_x = 0$ and $\sigma_{xy} = \sigma_{yx}$ for moment equilibrium.

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0$$

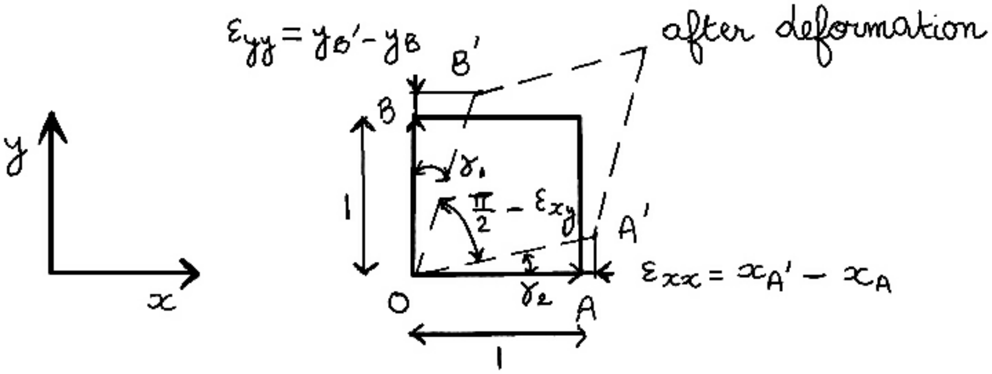
Similarly in 3-D, $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = 0$ and $\sigma_{xy} = \sigma_{yx}$ for moment equilibrium.

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = 0 \qquad \sigma_{xz} = \sigma_{zx}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \qquad \sigma_{zy} = \sigma_{yz}$$

3) Strain at a point

Consider a square element whose sides are of unit length. Under the action of the external loading, the element will deform such that the sides of the square are no longer perpendicular.



$\varepsilon_{xy} = \gamma_1 + \gamma_2 > 0$ when angle $B'OA'$ becomes smaller than $\frac{\pi}{2}$. Strain is dimensionless.

4) Strain-displacement relations

For small deformations strains and displacements are related as follows:

In 2-D, $\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

where u, v are the displacements of a point in the x and y directions.

In 3-D,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad \varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

where u, v, w are the displacements of a point in the x, y, z directions.

In matrix form, $\{\varepsilon\} = [L]\{U\}$, where,

- in 2-D, $\{\varepsilon\}^T = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}]$, $[L] = \begin{bmatrix} \frac{\partial \cdot}{\partial x} & 0 \\ 0 & \frac{\partial \cdot}{\partial y} \\ \frac{\partial \cdot}{\partial y} & \frac{\partial \cdot}{\partial x} \end{bmatrix}$ and $\{U\}^T = [u, v]$
- in 3-D, $\{\varepsilon\}^T = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}]$, $[L] = \begin{bmatrix} \frac{\partial \cdot}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \cdot}{\partial y} & 0 \\ 0 & 0 & \frac{\partial \cdot}{\partial z} \\ \frac{\partial \cdot}{\partial y} & \frac{\partial \cdot}{\partial x} & 0 \\ 0 & \frac{\partial \cdot}{\partial z} & \frac{\partial \cdot}{\partial y} \\ \frac{\partial \cdot}{\partial z} & 0 & \frac{\partial \cdot}{\partial x} \end{bmatrix}$ and $\{U\}^T = [u, v, w]$

5) Compatibility equations

In 2-D, since $\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ and $\varepsilon_{yy} = \frac{\partial v}{\partial y}$,
$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \partial u}{\partial x \partial y^2} + \frac{\partial^2 \partial v}{\partial x^2 \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$, hence an extra relationship between three strains and two displacements.

Similar compatibility equations are obtained in 3-D. Compatibility equations are automatically satisfied in the stiffness approach used in the displacement-based finite element method described in this course.

6) A constitutive relationship - Hooke's law

The uniaxial Hooke's law $\sigma = E\varepsilon$, where E is the elastic modulus, can be generalized , and is expressed in 3-D as $\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}$ where $[D]$ is the material property matrix, $\{\sigma\}$ the stress vector, $\{\varepsilon\}$ the strain vector, $\{\varepsilon_0\}$ the initial strain vector, and $\{\sigma_0\}$ the initial (or residual) stress vector.

$\{\sigma\}^T = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}]$
$\{\varepsilon\}^T = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}]$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

E : elastic modulus, ν : Poisson's ratio

Other specific $[D]$ matrices will be found for plane stress, plane strain, or axisymmetric analyses. $[D]$ is symmetric for both isotropic and anisotropic materials. In the isotropic case, only E and ν are needed. $\{\sigma_0\}$ represents stresses that are known to exist in a material before it is loaded. They must be specified by the analyst. $\{\varepsilon_0\}$ may be the result of crystal growth, shrinkage, or temperature changes.

Note that in different textbooks, shear stresses are often noted with τ (e.g. σ_{xy} is noted τ_{xy}), and shear strains are often noted with γ (e.g. ε_{xy} is noted γ_{xy}).

7) Principle of minimum potential energy

How to find the mechanical equilibrium of a structure other than by writing the equilibrium equations for each component of the structure, and hoping to be able to solve for all of them? The principle of minimum potential energy (presented in this section, and which you've encountered many times in physics) and the principle of virtual displacements (presented in next section) are powerful tools to look for the equilibrium of a whole structure at once.

The principle of minimum potential energy (PMPE) states that: "out of all the possible displacements fields that satisfy the geometric boundary conditions (i.e. prescribed displacements), the one that also satisfies the equations of static equilibrium results in the minimum of total potential energy for the structure (or body)."

The total potential energy Π is defined as the sum of the strain energy (internal potential energy U_i) and the external potential energy U_e from the external forces. $\Pi = U_i + U_e$. For conservative systems, the loss of external potential energy during the loading process must be equal to the work W_e done on the system by the external forces, or $-U_e = W_e$, and therefore $\Pi = U_i - W_e$. Π is a function of functions (strains and displacements) and is called a functional. Minimizing Π is called a variational problem. The first variation of the total potential energy $\delta\Pi = \delta U_i - \delta W_e$ must be zero, i.e. $\delta U_i = \delta W_e$.

In a global Cartesian (x, y, z) coordinate system,
$\delta U_i = \int_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \sigma_{xy}\delta\varepsilon_{xy} + \sigma_{yz}\delta\varepsilon_{yz} + \sigma_{zx}\delta\varepsilon_{zx})dV.$

Using matrix notation, $\delta U_i = \int_V \{\delta\varepsilon\}^T\{\sigma\}dV$.

For the work of external forces, considering a body force $\{b\}$ (per unit volume), a surface traction $\{s\}$ (per unit area) and N point loads $\{f_p\}$, then

$$\delta W_e = \int_V (b_x\delta u + b_y\delta v + b_z\delta w)dV + \int_A (s_x\delta u + s_y\delta v + s_z\delta w)dA + \sum_{p=1}^{p=N}(f_{px}\delta u + f_{py}\delta v + f_{pz}\delta w).$$

Using matrix notation, $\delta W_e = \int_V \{\delta U\}^T\{b\}dV + \int_A \{\delta U\}^T\{s\}dA + \sum_{p=1}^{p=N}\{\delta U\}^T\{f_p\}$, where

$\{b\}^T = [b_x, b_y, b_z]$: body force vector
$\{s\}^T = [s_x, s_y, s_z]$: surface traction vector
$\{f_p\}^T = [f_{px}, f_{py}, f_{pz}]$: point load vector
$\{\delta U\}^T = [\delta u, \delta v, \delta w]$: first variation of displacement vector

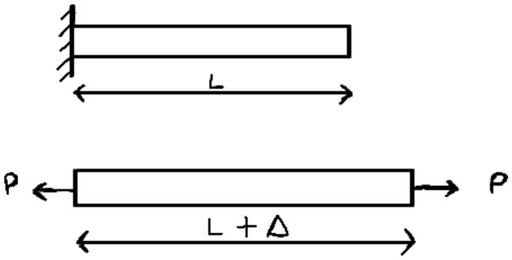
Eliminating $\{\sigma\}$ using the linear elastic stress-strain relationship, the PMPE becomes

$$\int_V \{\delta\varepsilon\}^T[D]\{\varepsilon\}dV =$$
$$\int_V \{\delta\varepsilon\}^T[D]\{\varepsilon_0\}dV - \int_V \{\delta\varepsilon\}^T\{\sigma_0\}dV + \int_V \{\delta U\}^T\{b\}dV + \int_A \{\delta U\}^T\{s\}dA + \sum_{p=1}^{p=N}\{\delta U\}^T\{f_p\}.$$

Going back to the total potential energy, it is given by

$$\Pi = \frac{1}{2}\int_V \{\varepsilon\}^T[D]\{\varepsilon\}dV - \int_V \{\varepsilon\}^T[D]\{\varepsilon_0\}dV + \int_V \{\varepsilon\}^T\{\sigma_0\}dV$$
$$- \int_V \{U\}^T\{b\}dV - \int_A \{U\}^T\{s\}dA - \sum_{p=1}^{p=N}\{U\}^T\{f_p\}$$

Example: consider the elongation Δ and an axial force P in a uniaxial stress member (bar) of uniform cross-sectional area A, length L and modulus of elasticity E. One end of the bar is fixed.



$\varepsilon_0 = 0,\ \sigma_0 = 0,\ b = 0$ and $s = 0$.
$\Pi = \frac{1}{2}E\varepsilon^2AL - P\Delta$ with $\varepsilon = \frac{\Delta}{L}$, therefore $\Pi = \frac{1}{2}E(\frac{\Delta}{L})^2AL - P\Delta$.
We want Δ for equilibrium, i.e. Δ for minimum Π.
$\frac{\delta\Pi}{\delta\Delta} = E\frac{\Delta}{L^2}AL - P = 0 \Leftrightarrow \Delta = \frac{PL}{AE}$: this is well known!
Note that $\frac{\delta^2\Pi}{\delta\Delta^2} = \frac{EA}{E} > 0$, therefore, Δ is really at a minimum of Π.

8) Principle of virtual displacements (or virtual work)

This principle will be very convenient and useful for finite element formulation of complex problems. Work is the product of a displacement and the component of the force in the direction of the displacement. Virtual work is imagined to occur when the forces are real and the displacements are virtual (imagined), or vice-versa, but this not used herein.

Statement of the principle of virtual displacements (PVD): if the work done by the external forces on the structure is equal to the increase in strain energy for any set of admissible virtual displacements (i.e. satisfying the prescribed displacements), then the system is in equilibrium.

Let us denote the virtual displacements in x, y, z directions as $\delta u, \delta v, \delta w$ (not variations!). The virtual displacements will cause virtual strains $\delta\varepsilon_{xx}, \delta\varepsilon_{yy}, \delta\varepsilon_{zz}, \delta\varepsilon_{xy}, \delta\varepsilon_{yz}, \delta\varepsilon_{zx}$.

In matrix form, the PVD becomes
$$\int_V \{\delta\varepsilon\}^T\{\sigma\}dV = \int_V \{\delta U\}^T\{b\}dV + \int_A \{\delta U\}^T\{s\}dA + \sum_{p=1}^{p=N}\{\delta U\}^T\{f_p\} \qquad \forall\{\delta U\}, \{\delta\varepsilon\}$$
with notations as in the previous section.
This is known as a weak form of the equilibrium equations because this equation only contains first derivatives of displacements whereas the original equilibrium equations (see section 2 + Hooke's law) contain second order derivatives of the displacements.