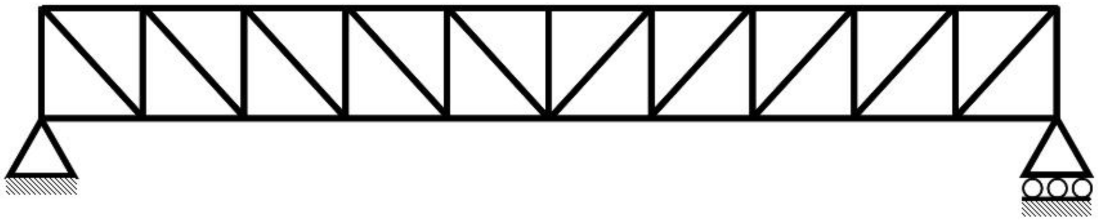
Introduction to FEM-The direct approach-Trusses

Trusses: structures composed of straight members connected at joints by pins. Most or all members in a truss do not experience bending or torsional moments.

Example:



Given: external forces, geometry, material properties.

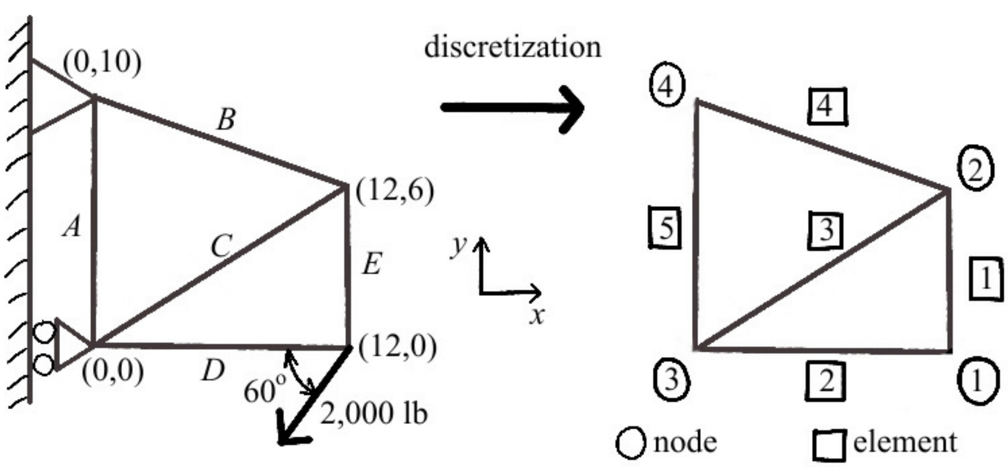
Unknown: displacements at each joint; axial elongation, strain, stress, force for each member.

1) Discretization

Let us consider each member as an element:

Element e has two nodes i and j at either end. Forces are transmitted from one element to the next at nodes. ***Bar elements*** can only take axial forces (***beam elements*** allow bending moments). i and j (lowercase) are the local node numbers within the element. I and J (uppercase) denote global node numbers in the whole structure. Elements are also numbered.

Example:



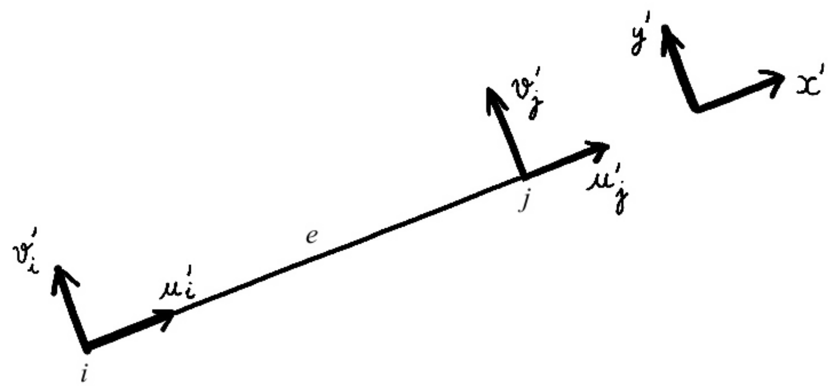
Nodal coordinates:	Node number	x coordinate (in)	y coordinate (in)
	1	12	0
	2	12	6
	3	0	0
	4	0	10

Element data (connectivity table):

Element	Node i	Node j	Material flag
1	1	2	1 : 0.5-in (diameter) steel
2	3	1	2 : 0.4-in aluminium
3	3	2	1
4	4	2	2
5	3	4	1

2) Element stiffness relationship in local coordinates

Let (x', y') be the local coordinate system for element e, with x' along the length of element from i to j, and y' perpendicular to x'.



The nodal displacements are noted as u_i' and v_i' in x' and y', respectively, at node i. The corresponding forces are noted F_{xi}' and F_{yi}'. From elementary strength of materials, $\delta = \frac{PL}{AE}$, where δ is the axial elongation, L the member length, P the axial force, A the cross-sectional area and E the elastic modulus. It is assumed that the elastic range is not exceeded and that A is constant. In other words,

$$F_{xi}' = \frac{AE}{L}(u_i' - u_j')$$
$$F_{xj}' = \frac{AE}{L}(u_j' - u_i')$$

where $F_{xi}' = -F_{xj}'$ for equilibrium.

Because bar elements do not withstand transverse forces, $F_{yi}' = F_{yj}' = 0.$

In matrix form,

$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix} = \begin{Bmatrix} F_{xi}' \\ F_{yi}' \\ F_{xj}' \\ F_{yj}' \end{Bmatrix}$$

With global nodal coordinates as (x_i, y_i) and (x_j, y_j) for nodes i and j, the element length can be computed as $L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$. The other two properties A and E can be specified for each element. Concisely,
$\left[K^{e'}\right]\left\{U^{e'}\right\} = \left\{F^{e'}\right\}$, where $\left[K^{e'}\right]$ represents the local element stiffness matrix, $\left\{U^{e'}\right\}$ the local nodal displacement vector, and $\left\{F^{e'}\right\}$ the local element nodal force vector. "e" denotes element, " $'$ " denotes local coordinate system.

Example*:* for element 3:

$$\left[K^{(3)'}\right] = \frac{A^{(3)}E^{(3)}}{L^{(3)}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Material flag set to 1: 0.50-in steel: $A^{(3)} = \frac{\pi}{4}(0.5)^2 = 0.196\,\text{in}^2$.

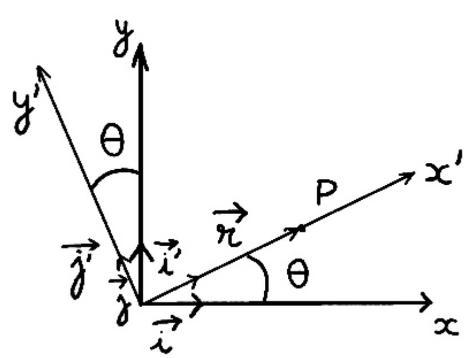
$$E^{(3)} = 30 \times 10^6 \text{ psi}$$

$$L^{(3)} = \sqrt{(0-12)^2 + (0-6)^2} = 13.42\,\text{in}$$

Then, $\left[K^{(3)'}\right] = 10^3\begin{bmatrix} 438 & 0 & -438 & 0 \\ 0 & 0 & 0 & 0 \\ -438 & 0 & 438 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ lbf/in

3) Transformation from local to global coordinates

In the global (x, y) coordinate system, position vector $\vec{r}$ to an arbitrary point P can be written as $\vec{r} = r_x\vec{i} + r_y\vec{j}$. In the bar (rotated) coordinate system (x', y'), $\vec{r} = r_{x'}\vec{i'} + r_{y'}\vec{j'}$.



Scalar product: "." : $\vec{u}.\vec{v} = \|\vec{u}\|\ \|\vec{v}\|\cos(\vec{u},\vec{v})$

$$\begin{aligned} \vec{r}.\vec{i} &= r_x\vec{i}.\vec{i} + r_y\vec{j}.\vec{i} &&= r_{x'}\vec{i'}.\vec{i} + r_{y'}\vec{j'}.\vec{i} \\ &= r_x + 0 &&= r_{x'}\cos(x', x) + r_{y'}\cos(y', x) \end{aligned}$$

$$\vec{r}.\vec{j} = r_x\vec{i}.\vec{j} + r_y\vec{j}.\vec{j} = r_{x'}\vec{i'}.\vec{j} + r_{y'}\vec{j'}.\vec{j}$$
$$= 0 + r_y \qquad = r_{x'}\cos(x',y) + r_{y'}\cos(y',y)$$

In matrix form,

$$\begin{Bmatrix} r_x \\ r_y \end{Bmatrix} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} \begin{Bmatrix} r_{x'} \\ r_{y'} \end{Bmatrix} \text{ or } \{r\} = [T]\{r'\} \quad \text{with}$$

$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} \cos(x',x) & \cos(y',x) \\ \cos(x',y) & \cos(y',y) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(\theta+\frac{\pi}{2}) = -\sin\theta \\ \cos(\theta-\frac{\pi}{2}) = \sin\theta & \cos\theta \end{bmatrix}.$$

It can be shown that $[T]$ is orthogonal, i.e. $[T]^{-1} = [T]^T$, therefore $\{r'\} = [T]^T\{r\}$.

4) Global element stiffness relationship

Interpreting $\begin{Bmatrix} r_{x'} \\ r_{y'} \end{Bmatrix}$ as $\begin{Bmatrix} u'_i \\ v'_i \end{Bmatrix}$ and $\begin{Bmatrix} r_x \\ r_y \end{Bmatrix}$ as $\begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$, we get

$$\begin{Bmatrix} u'_i \\ v'_i \\ u'_j \\ v'_j \end{Bmatrix} = \begin{bmatrix} n_{11} & n_{21} & 0 & 0 \\ n_{12} & n_{22} & 0 & 0 \\ 0 & 0 & n_{11} & n_{21} \\ 0 & 0 & n_{12} & n_{22} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

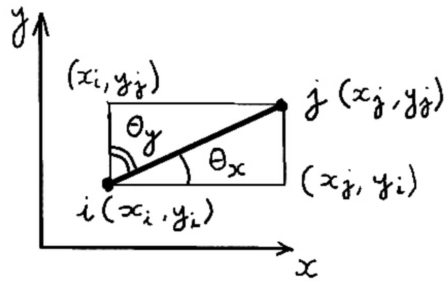
or more concisely, $\{U^{e'}\} = [R]\{U^e\}$, where $[R] = \begin{bmatrix} T^T & 0 \\ 0 & T^T \end{bmatrix}$

and $\{U^e\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$. Similarly, $\{F^{e'}\} = [R]\{F^e\}$, with $\{F^e\} = \begin{Bmatrix} F_{xi} \\ F_{yi} \\ F_{xj} \\ F_{yj} \end{Bmatrix}$.

$[K^{e'}]\{U^{e'}\} = \{F^{e'}\}$ becomes $[K^{e'}][R]\{U^e\} = [R]\{F^e\}$, or
$[R]^T[K^{e'}][R]\{U^e\} = \{F^e\}$, since $[R]^T[R] = [R]^{-1}[R] = [I]$.

More simply, $[K^e]\{U^e\} = \{F^e\}$, where $[K^e] = [R]^T[K^{e'}][R]$ is the global element stiffness matrix

$$[K^e] = \frac{AE}{L}\begin{bmatrix} n_{11}^2 & n_{21}n_{11} & -n_{11}^2 & -n_{21}n_{11} \\ n_{11}n_{21} & n_{21}^2 & -n_{11}n_{21} & -n_{21}^2 \\ -n_{11}^2 & -n_{21}n_{11} & n_{11}^2 & n_{21}n_{11} \\ -n_{11}n_{21} & -n_{21}^2 & n_{11}n_{21} & n_{21}^2 \end{bmatrix} \text{ with } \begin{matrix} n_{11} = \cos\theta_x = \cos(x,x') = \frac{x_j-x_i}{L} \\ n_{21} = \cos\theta_y = \cos(x',y) = \frac{y_j-y_i}{L} \end{matrix} .$$



Example: for element 3:

$n_{11} = \frac{x_2 - x_3}{L^{(3)}} = \frac{12-0}{13.42} = 0.8944$

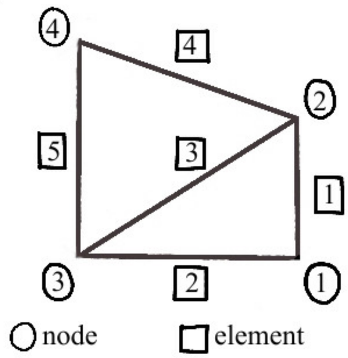
$n_{21} = \frac{y_2 - y_3}{L^{(3)}} = \frac{6-0}{13.42} = 0.4472$

$$[K^{(3)}] = 10^3 \begin{bmatrix} 351 & 176 & -351 & -176 \\ 176 & 88 & -176 & -88 \\ -351 & -176 & 351 & 176 \\ -176 & -88 & 176 & 88 \end{bmatrix} \text{lbf/in}$$

5) Assemblage

The original structure is put back together from individual elements. This is done based on compatibility: the x- and y- displacements at one node must be identical to those of the other nodes from other elements to be merged.

Example: for the truss below, determine the assemblage stiffness matrix in terms of the 2x2 global stiffness submatrices $[K^e_{I,J}]$.



Let us first create a null assemblage stiffness matrix $[K^a]$ involving global node numbers 1 to 4.

$$
\begin{array}{c|c}
 & \begin{array}{cccc} \quad 1 & \quad 2 & \quad 3 & \quad 4 \end{array} \\
[K^a] = & \left[\begin{array}{cccc} 0_{2x2} & 0_{2x2} & 0_{2x2} & 0_{2x2} \\ 0_{2x2} & 0_{2x2} & 0_{2x2} & 0_{2x2} \\ 0_{2x2} & 0_{2x2} & 0_{2x2} & 0_{2x2} \\ 0_{2x2} & 0_{2x2} & 0_{2x2} & 0_{2x2} \end{array} \right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

From the connectivity table, element 1 has global node numbers 1 and 2, so

$$[K^{(1)}] = \begin{bmatrix} K_{1,1}^{(1)} & K_{1,2}^{(1)} \\ K_{2,1}^{(1)} & K_{2,2}^{(1)} \end{bmatrix} \qquad \Delta = 1$$

Element 2 has global node numbers 3 and 1, so

$$[K^{(2)}] = \begin{bmatrix} K_{3,3}^{(2)} & K_{3,1}^{(2)} \\ K_{1,3}^{(2)} & K_{1,1}^{(2)} \end{bmatrix} \qquad \Delta = 2$$

Element 3 has global node numbers 3 and 2, so

$$[K^{(3)}] = \begin{bmatrix} K_{3,3}^{(3)} & K_{3,2}^{(3)} \\ K_{2,3}^{(3)} & K_{2,2}^{(3)} \end{bmatrix} \qquad \Delta = 1$$

Element 4 has global node numbers 4 and 2, so

$$[K^{(4)}] = \begin{bmatrix} K_{4,4}^{(4)} & K_{4,2}^{(4)} \\ K_{2,4}^{(4)} & K_{2,2}^{(4)} \end{bmatrix} \qquad \Delta = 2$$

Element 5 has global node numbers 3 and 4, so

$$[K^{(5)}] = \begin{bmatrix} K_{3,3}^{(5)} & K_{3,4}^{(5)} \\ K_{4,3}^{(5)} & K_{4,4}^{(5)} \end{bmatrix} \qquad \Delta = 1$$

$$\Delta_{\max} = 2$$

Finally,

$$
\begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & \\
[K^a] = & \left[\begin{array}{cccc}
K_{1,1}^{(1)} + K_{1,1}^{(2)} & K_{1,2}^{(1)} & K_{1,3}^{(2)} & 0_{2x2} \\
K_{2,1}^{(1)} & K_{2,2}^{(1)} + K_{2,2}^{(3)} + K_{2,2}^{(4)} & K_{2,3}^{(3)} & K_{2,4}^{(4)} \\
K_{3,1}^{(2)} & K_{3,2}^{(3)} & K_{3,3}^{(2)} + K_{3,3}^{(3)} + K_{3,3}^{(5)} & K_{3,4}^{(5)} \\
0_{2x2} & K_{4,2}^{(4)} & K_{4,3}^{(5)} & K_{4,4}^{(4)} + K_{4,4}^{(5)}
\end{array}\right] & & & & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

Note that each individual 2x2 element stiffness submatrix is symmetric, and $[K^a]$ is symmetric.

$[K^a]$ is also banded, i.e. the nonzero entries are gathered along the main diagonal. This is important numerically as special solvers for banded matrices exist and run faster than for full matrices. The speed of execution is also related to the size of the half-bandwidth $b_w = N_{DOF} \times (1 + \Delta_{\max})$, where N_{DOF} is the number of degrees of freedom per node and $\Delta_{\max}$ is the maximum difference between node numbers in any element of the whole structure. Proper node numbering is therefore theoretically important to keep b_w small. In the example above, $b_w = 2 \times (1 + 2) = 6.$ Note that in commercial codes, internal renumbering is done automatically, if necessary, in order to minimize b_w.

6) Application of loads

By design in a truss, the external loads can only occur at the joints (nodes).

Example*:* in the truss above, the x- and y- components of the applied load are $F_x = -2,000\cos 60° = -1,000\,\text{lbf}$ and $F_y = -2,000\sin 60° = -1,732\,\text{lbf}$. The load is applied to node 1. There is an unknown reaction force in the x-direction at node 3 (roller) and two unknown reaction forces in the x- and y-directions at node 4 (pin).

$$
\{F^a\} = \left\{\begin{array}{c} -1,000 \\ -1,732 \\ 0 \\ 0 \\ R_x^3 \\ 0 \\ R_x^4 \\ R_y^4 \end{array}\right\}
\begin{array}{ll} F_x & \text{Node 1} \\ F_y & \\ F_x & \text{Node 2} \\ F_y & \\ F_x & \text{Node 3} \\ F_y & \\ F_x & \text{Node 4} \\ F_y & \end{array}
$$

7) Application of restraint on nodal displacements and solution

After assemblage, the system equation is of the form

$[K^a]\{U^a\} = \{F^a\}$, where $\{U^a\}$ is the vector of nodal displacements.

$\{U^a\}^T = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad \ldots \quad u_N \quad v_N\}^T$, where N is the maximum number of nodes and $u_I,\ v_I$ are the x and y displacements at global node number $I.$ For now, no restraint on the nodal displacements have been considered, and the whole structure can fly into space!!! This is a rigid body motion, and $\{U\}$ cannot be determined because $[K^a]$ cannot be inverted (singular matrix, i.e. its determinant is zero). After the restraints are considered and enforced, the system equation becomes $[K]\{U\} = \{F\}$, where $[K]$ is non-singular, therefore $[K]^{-1}$ exists, which implies a unique solution for $\{U\} = [K]^{-1}\{F\}$.

Methods for enforcing displacement restraints:

Method 1 (interesting for a technique called sub-structuring, but not used otherwise)

Consider the system
$$\begin{aligned} k_{11}a_1 + k_{12}a_2 + k_{13}a_3 &= f_1 \\ k_{21}a_1 + k_{22}a_2 + k_{23}a_3 &= f_2 \\ k_{31}a_1 + k_{32}a_2 + k_{33}a_3 &= f_3 \end{aligned}$$
with $k_{ij} = k_{ji}$

Let us impose $a_2 = a_{\text{known}}$, therefore the set of equations above is equivalent to e.g.
$$\begin{aligned} k_{11}a_1 + k_{12}a_2 + k_{13}a_3 &= f_1 \\ a_2 &= a_{\text{known}} \\ k_{31}a_1 + k_{32}a_2 + k_{33}a_3 &= f_3 \end{aligned}$$
The symmetry has been destroyed but can be restored by transposing terms involving a_2 to the right-hand side, and with a_{known} replacing a_2 :
$$\begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & 1 & 0 \\ k_{31} & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} f_1 - k_{12}a_{\text{known}} \\ a_{\text{known}} \\ f_3 - k_{32}a_{\text{known}} \end{Bmatrix}.$$
Note that f_2 disappears: a known displacement leads to an unknown reaction force.

Method 2 (interesting for programming of the finite element method)

Consider the same initial system as above. Let us select a large number β (6 to 12 orders of magnitude larger than the largest coefficient k_{ij}). β is added to k_{ii} if a_i is prescribed, and the right-hand side of the i –th equation is changed to β times the prescribed value. In matrix form, the system becomes
$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & (k_{22}+\beta) & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \beta \times a_{\text{known}} \\ f_3 \end{Bmatrix}.$$
Symmetry is preserved, and this method is very easy to implement in a computer program. For practical purposes, $a_2 = a_{\text{known}}$.

Method 3 (the only one to use when doing hand calculations)

When a displacement is known to be zero from the boundary conditions, then the initial system equation $[K^a]\{U^a\} = \{F^a\}$ can be rewritten into $[K]\{U\} = \{F\}$ by simply removing the columns and rows corresponding to that displacement.

8) Computation of element resultants

Element resultants include axial elongation, strains, stresses and forces. All these are computed from the nodal displacements. For element $e,$ elongation $\delta = u'_j - u'_i,$ where $u'_j = n_{11}u_j + n_{21}v_j$ and $u'_i = n_{11}u_i + n_{21}v_i.$

At this stage, u_i, v_i, u_j, v_j and n_{11}, n_{21} are known therefore δ is known. The axial strain $\varepsilon = \frac{\delta}{L}$ can be computed. The material is assumed to be elastic, therefore, $\sigma = E\varepsilon$ is known, and the axial force $F = \sigma A$ can be determined. Note that for $\delta, \varepsilon, \sigma$ and $F,$ positive values denote tension, while negative values denote compression. Make sure to check that these signs make physical sense.

9) Examples

Assignment 1.

For additional problems to practice with, use the results in the statement of Assigment 1, and find yourself $[K^{(2)}]$, $[K^{(4)}]$ and $[K^{(5)}]$.