1. Which of the following subsets of $\mathbf{R}^4$ are closed under (the standard operation of) multiplication by scalars?

A. $\{(a,\ b,\ c,\ d) |\ a\, b\, c = 0\}$
B. $\{(a,\ b,\ c,\ d) |\ a = 1, b = 0 \text{ and } c + d = 0\}$
C. $\{(a,\ b,\ c,\ d) |\ a > 1 \text{ and } b < 1\}$
D. $\{(a,\ b,\ c,\ d) |\ a > 0 \text{ and } b > 0\}$
E. $\{(a,\ b,\ c,\ d) |\ a - b + 2c = 0\}$

Recall for $s \in \mathbb{R}$ we have $s(a,b,c,d) = (sa, sb, sc, sd)$

(A) If $abc = 0$ then $(sa)(sb)(sc) = s^3 abc = 0$, so (A) is closed

(B) If $a = 1$, then $sa \neq 1$ in general, e.g. for $s = 2$, so (B) is not closed

(C) If $a > 1$, then $sa > 1$ in general, e.g. for $s = -1$, so (C) is not closed.

(D) Not closed, same argument as in (C)

(E) $(sa) - (sb) + 2(sc) = s(a - b + 2c) = 0$, so (E) is closed.

ANSWER $\boxed{A, E}$

2. Which of the following are subspaces of $\mathbf{F}[\mathbf{R}] = \{f \mid f\colon \mathbf{R} \to \mathbf{R}\}$?

$U = \{f \in \mathbf{F}[\mathbf{R}] \mid f(-1)f(1) = 0\}$
$V = \{f \in \mathbf{F}[\mathbf{R}] \mid f(1) + f(2) = 0\}$
$S = \{f \in \mathbf{F}[\mathbf{R}] \mid f(x) = f(-x),\ \forall x \in \mathbf{R}\}$
$T = \{f \in \mathbf{F}[\mathbf{R}] \mid f(1) \leq 0\}$

$\boxed{U}$: not closed under addition: Let $f \in F[\mathbb{R}]$ be given by $f(x) = x + 1$. Then $f(-1) = 0$, so $f \in U$. Also, consider the function $g \in F[\mathbb{R}]$, $g(x) = x - 1$. Then $g \in U$ since $g(1) = 0$ so $g(1)g(-1) = 0$. But $(f+g)(-1) = f(-1) + g(-1) = 0 - 2 = -2$, $(f+g)(1) = f(1) + 0 = 2 + 0 = 2$, so $(f+g)(1)(f+g)(-1) = (-2)(2) = -4 \neq 0$, and $f + g \notin U$.

$\boxed{V}$ is a subspace (apply subspace test)

$\boxed{S}$ yes, is a subspace (apply subspace test)

$\boxed{T}$ no, not closed under scalar multiplication: Let $f \in F[\mathbb{R}]$, $f(x) = -1$ (constant function!), so $f \in T$. But $-2f$ is the constant function $(-2)(-1) = 2$, so $-2f \notin T$.

ANSWER $\boxed{V, S}$

Question 2)

V is a subspace. We check the conditions of the subspace test

- $0_{\mathbb{F}[\mathbb{R}]} \in V$, since $(0_{\mathbb{F}[\mathbb{R}]})(1) + (0_{\mathbb{F}[\mathbb{R}]})(2) = 0 + 0 = 0$
- Suppose $f, g \in V$, i.e., $f(1) + f(2) = 0 = g(1) + g(2)$. But then also $(f+g)(1) + (f+g)(2) = f(1) + g(1) + f(2) + g(2) = 0 + 0 = 0$
- Suppose $f \in V$, and $a \in \mathbb{R}$. Then $(af)(1) + (af)(2) = af(1) + af(2) = a(f(1) + f(2)) = a \cdot 0 = 0$

S is a subspace. We check the conditions of the subspace test:

- $0_{\mathbb{F}[\mathbb{R}]} \in S$ since for all $x \in \mathbb{R}$ we have $0_{\mathbb{F}[\mathbb{R}]}(x) = 0 = 0_{\mathbb{F}[\mathbb{R}]}(-x)$.
- Suppose $f, g \in S$. Then $(f+g)(x) = f(x) + g(x) \overset{f,g \in S}{=} f(-x) + g(-x) = (f+g)(-x)$, so $f+g \in S$.
- Suppose $f \in S$, $a \in \mathbb{R}$. Then $(af)(x) = af(x) \overset{f \in S}{=} af(-x) = (af)(-x)$, so $af \in S$.

two

~~three~~

3. Which ~~two~~ of the following statements are true?

I. The span of any vector in $\mathbf{R}^3$ is a line through the origin.

II. The span of any two distinct vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$.

III. A set of vectors $\{u, v, w\}$ in a vector space spans V if every vector on V is a linear combination of $u, u+v$ and $u+v+w$.

~~IV. Any spanning set for $\mathbf{M}_{2,2}$ contains at least four elements.~~

V. The set $\{(1,1),(2,3)\}$ spans $\mathbf{R}^2$.

(I) False: The span of $0_{\mathbb{R}^3}$ is $\{0_{\mathbb{R}^3}\}$.

(II) False: The span of $0_{\mathbb{R}^2}$ and a $0 \neq d \in \mathbb{R}^2$ is the line through d.

(III) True since every $x \in V$ has the form $x = au + b(u+v) + c(u+v+w)$
$= (a+b+c)u + (b+c)v + cw$ for $a, b, c \in \mathbb{R}$.

(IV) True by the Fundamental Theorem

(V) True;

$(a,b) = (3a-2b)(1,1) + (b-a)(1,1)$

ANSWER $\boxed{\text{III}, \cancel{\text{IV}}\ \text{V}}$

4. For which value of s does the vector (6, 3, s) belong to the subspace of $\mathbf{R}^3$ spanned by $(1,2,3)$ and $(0,1,2)$?

We need to find $x, y \in \mathbb{R}$ such that

$x(1,2,3) + y(0,1,2) = (6,3,s)$.

Equivalently, for which s is $\left\{\begin{array}{ll} x & = 6 \\ 2x + y & = 3 \\ 3x + 2y & = s \end{array}\right\}$ solvable.

We get $x = 6$, $y = 3 - 2x = 3 - 12 = -9$, so $s = 3x + 2y = 3 \cdot 6 + 2 \cdot (-9) =$
$= 18 - 18 = 0$

ANSWER $\boxed{s = 0}$

5. If we give $X = \mathbf{R}^2$ the *non-standard* operations

$$(x,\ y) \oplus (x',\ y') = (x + x' - 1,\ y + y' + 2) \quad \text{(vector addition)}$$

and

$$k \odot (x,\ y) = (kx - k + 1,\ ky + 2k - 2) \quad \text{(multiplication by scalars),}$$

then X *is* a real vector space.

- What is the zero vector of X?
- If $\mathbf{v} = (x,\ y)$ is in X then what is $-\mathbf{v}$?

ANSWER $\boxed{0 = (1, -2) \qquad -\mathbf{v} = (-x+2, -y-4)}$

Let $0 = (x', y')$. Then $(x,y) \oplus (x+x'-1, y+y'+2) = (x,y)$

so $\left\{ \begin{array}{l} x + x' - 1 = x \\ y + y' + 2 = y \end{array} \right\} \Leftrightarrow \begin{array}{l} x' = 1 \\ y' = -2 \end{array}$

$-v = (-1)\odot(v)$, ~~in general~~. So, if $v = (x,y)$ we get

$-v = (-1)\odot = ((-1)x - (-1) + 1,\ (-1)y + 2(-1) - 2)$

$= (-x + 1 + 1,\ -y - 2 - 2) = (-x+2, -y-4)$

Totals for each part

6. Let $W = \{(x,y,z) \in \mathbf{R}^3 \mid x - y + z = 0\}$

2 a) Is W a subspace of $\mathbf{R}^3$?

2.5 b) Find a spanning set for W.

1.5 c) Give a complete geometric description of W.

(You must justify your answers.)

(a) Subspace test:

- $0_{\mathbb{R}^3} = (0,0,0) \in W$ since $\underset{x\text{-component of } 0_{\mathbb{R}^3}}{0} - \underset{y\text{-component of } 0_{\mathbb{R}^3}}{0} + \underset{z\text{-component of } 0_{\mathbb{R}^3}}{0} = 0$

1.5 points for justification

- Suppose $u = (x,y,z) \in W$ and $v = (x', y', z') \in W$. Then $u + v = (x+x', y+y', z+z')$, and $(x+x') - (y+y') + (z+z') = (x - y + z) + (x' - y' + z') = 0 + 0 \langle \text{using } u, v \in W \rangle = 0$ so $u + v \in W$.

.5

- Suppose $u = (x,y,z) \in W$ and $a \in \mathbb{R}$. Then $au = (ax, ay, az)$, $ax - ay + az = a(x - y + z) = a \cdot 0 = 0$, so $au \in W$.

Answer: Yes, W is a subspace.

(b) $(x,y,z) \in W \iff z = y - x$, so $W = \{(x,y,z) : z = y - x\} =$

$= \{(x, y, y-x) \in \mathbb{R}^3 : x, y \in \mathbb{R}\} = \{x(1,0,-1) + y(0,1,1) : x, y \in \mathbb{R}\}$

$= \text{span}\{(1,0,-1), (0,1,1)\}$

Any correct answer — 1

Hence $\{(1,0,-1), (0,1,1)\}$ is a spanning set.

plus 1.5 points for justification

NOTE: You can first solve (b), i.e. write W as a span, and then quote Theorem 4 in §5.1 (or Theorem 1 in §4.1) to conclude that W is a subspace, i.e. solve (a).

(c) W is a plane with normal $\vec{n} = (1,-1,1)$, through the origin.

.5 .5 .5

7. Consider the vector space $\mathbf{F}(\mathbf{R}) = \{f \mid f\colon \mathbf{R} \to \mathbf{R}\}$, with the standard operations. Recall that the zero of $\mathbf{F}(\mathbf{R})$ is the function that has the value 0 for all $x \in \mathbf{R}$.

Let $U = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) = f(-1)\}$ be the subspace of functions which have the same value at $x = -1$ and $x = 1$.

Define functions g, h, j and $k \in \mathbf{F}[\mathbf{R}]$ by

$$g(x) = 2x^3 - x^2 - 2x + 1, \quad h(x) = x^3 + x^2 - x + 1,$$
$$k(x) = -x^3 + 5x^2 + x + 1 \quad \text{and} \quad j(x) = x^3 - x, \qquad \forall x \in \mathbf{R}.$$

1 a) Show that g and h belong to U. (note $g(x) = 2x^3 - x^2 - 2x + 1$)

1.5 b) Show that $k \in \operatorname{span}\{g, h\}$.

2 c) Show that $j \notin \operatorname{span}\{g, h\}$.

1.5 d) Show that $\operatorname{span}\{g, h\} \neq \operatorname{span}\{g, h, j\}$.

(You must justify your answers.)

(a) $g(1) = 2\cdot 1 - 1 - 2\cdot 1 + 1 = 0$, $g(-1) = 2\cdot(-1) - 1 - 2(-1) + 1 = -2 - 1 + 2 + 1 = 0$

1

$h(1) = 1 + 1 - 1 + 1 = 2$, $h(-1) = -1 + 1 + 1 + 1 = 2$

so $g, h \in U$.

(b) We need to find $a, b \in \mathbb{R}$ satisfying $ag + bh = k$, ie

.5

$$\underbrace{a(2x^3 - x^2 - 2x + 1) + b(x^3 + x^2 - x + 1)} = -x^3 + 5x^2 + x + 1$$

(i.e. knowing what 'span means)

$$= (2a + b)x^3 + (b - a)x^2 + (-2a - b)x + (a + b)$$

This means

$2a + b = -1$, $b - a = 5$, $-2a - b = 1$, $a + b = 1$

.5

The solution to this system of linear equations is $a = -2$, $b = 3$. Indeed,

$$-2(2x^3 - x^2 - 2x + 1) + 3(x^3 + x^2 - x + 1) = -x^3 + 5x^2 + x + 1$$

Checking a, b work, or saying why matchcoefficients is sufficient

.5

(c) We know, by definition of the span, that

1

$j \in \text{span}\{g, h\} \iff$ there exist $a, b \in \mathbb{R}$ such that

knowing what to do

$$ag + bh = j$$

(see (b)!) $\iff$ there exist $a, b \in \mathbb{R}$ such that

$$(2a+b)x^3 + (b-a)x^2 + (-2a-b)x + (a+b) = x^3 - x$$

1

$\iff$ there exist $a, b \in \mathbb{R}$ such that

Any corrrect justification

(*) $2a+b = 1$, $b-a=0$, $-2a-b=-1$, $a+b=0$

But $b-a=0=a+b \iff a=0=b$, and then $2a+b = 2\cdot 0 + 0 = 0 \neq 1$. Hence the linear system (*) is NOT solvable, and therefore $j \notin \text{span}\{g, h\}$.

(d) $\text{span}\{g, h\} \neq \text{span}\{g, h, j\}$ since $j \notin \text{span}\{g, h\}$ while $j \in \text{span}\{g, h, j\}$, namely $j = 0g + 0h + 1 \cdot j$.

1.5 = 1- having some idea why two sets aren't the same,
+ .5 for doing it correctly in this case.

8. [Bonus] Let $\mathbf{E} = \{\text{"}ax + by + cz = d\text{"} \mid a, b, c, d \in \mathbf{R}\}$ be the set of linear equations with real coefficients in the variables x, y and z. Equip $\mathbf{E}$ with the usual operations on equations that you learned in high school: addition of equations, denoted here by "$\oplus$" and multiplication by scalars, denoted here by "$\circledast$", as follows:

$$\text{"}ax + by + cz = d\text{"} \oplus \text{"}ex + fy + gz = h\text{"} = \text{"}(a+e)x + (b+f)y + (c+g)z = d + h\text{"}$$

and

$$\forall k \in \mathbf{R}, \quad k \circledast \text{"}ax + by + cz = d\text{"} = \text{"}ka\,x + kb\,y + kc\,z = k\,d\text{"}.$$

You may assume without proof that $\mathbf{E}$ is a vector space.

1.5 points for a well-writtenjustification

Find a spanning set for $\mathbf{E}$.

(You must justify your answer.)

"ax + by + cz = d" = "ax = 0" ⊕ "by = 0" ⊕ "cz = 0" ⊕ "0 = d"

= a⊛ "x = 0" ⊕ b⊛ "y = 0" ⊕ c⊛ "z = 0" ⊕ d⊛ "0 = 1"

Hence $\mathbb{E}$ = span { "x=0", "y=0", "z=0", "0=1" }.

Thus { "x = 0", "y = 0", "z = 0", "0=1"} is a spanning set for E

1.5 points for a correct, well written answer