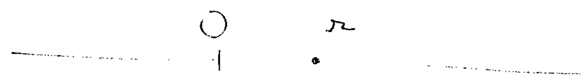
Review

2.5.1 Complex Numbers REVIEW

(R1)



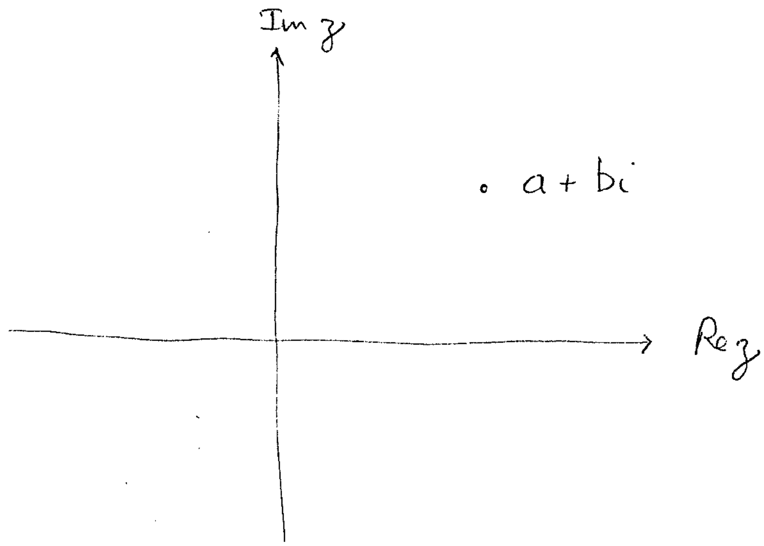
Real numbers ~~can't solve~~ $x^2+1=0$.

"Invent" i s.t

$i^2 = -1$.

Complex plane

Argand plane



"impossible #'s"

"i" - Euler 1777

$i = \sqrt{-1}$

Recall:

Defn $\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R},\ i^2 = -1 \}$ ← close bracket

set of all / things like this / such that / conditions on "things like this"

Set Notation

$z = a + bi \in \mathbb{C}$

"belongs to"

a = real part of z $\operatorname{Re}(z)$ b = imaginary part of z $\operatorname{Im}(z)$ (both are real numbers!)

($a = 0$: $z = bi$ pure imaginary) ($b = 0$ $z = a$ is real) $\mathbb{R} \subset \mathbb{C}$ "$\subset$" (subset of)

Equality $a + bi = c + di \iff a = c$ & $b = d$

Addition $(a+bi) + (c+di) = a + c + (b+d)i$

Mult^n $(a+bi)(c+di) = (ac - bd) + (ad + bc)i$

Rmk: They satisfy all the arithmetical properties of $\mathbb{R}$ except there is no ordering in $\mathbb{C}$. ie "$z > 0$" makes no sense.

e.g. solve $z(3+4i) = 1$, if possible.

Recall $(a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2$ ($\neq 0$ unless $a = b = 0$,

So, for $z = a + bi$, ~~def~~ let $\bar{z} = a - bi$ (complex conjugate of z)

$|z| = \sqrt{a^2+b^2}$ (modulus of z, absolute value, mod z)

Note: $|z| = 0 \iff z = 0$.

Then, $\boxed{\text{If } z \neq 0 \ (\Longleftrightarrow \text{ at least one of } a, b \text{ is not zero})}$, $|z| \neq 0$ so

$z \cdot \bar{z} = |z|^2$. so $z \cdot \dfrac{\bar{z}}{|z|^2} = 1$.

i.e. $z \neq 0 \Rightarrow \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$

Hence if $z(3+4i) = 1$, $z = \dfrac{1}{3+4i} = \dfrac{3-4i}{3^2+4^2} = \dfrac{3}{25} - \dfrac{4i}{25}$

L1 / L2

Other properties (easy to check)

$\overline{z+w} = \bar{z} + \bar{w}$

$\overline{zw} = \bar{z}\,\bar{w}$

$\overline{(\bar{z})} = z$

- $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$
- z pure im. $\Leftrightarrow z = -\bar{z}$

$|z| = |\bar{z}|$

$|zw| = |z|\,|w|$

$|z| \geq 0$, $|z| = 0 \Leftrightarrow z = 0$

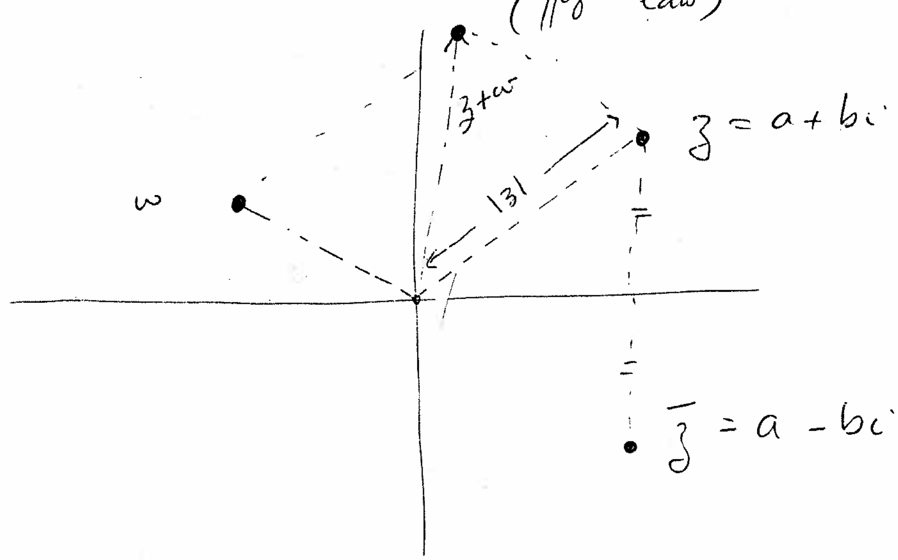
$|z+w| \leq |z| + |w|$

Δ inequality

L1 / L2 $\boxed{}$

back to Complex Plane : remind ourselves of the geom. interpretations of $z\bar{z}$, $|z|$. $z+w$

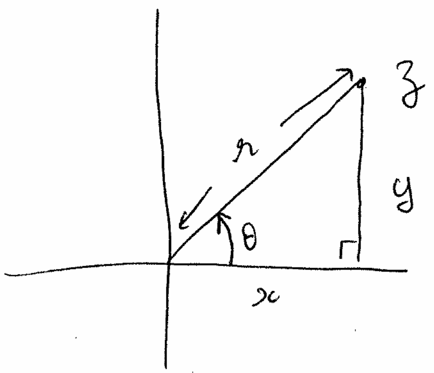
(||ogram law)



L2 R3

Polar form "$z = re^{i\theta}$" (polar coordinates)

- If $z = x+iy$; $x,y \in \mathbb{R}$ and r & θ are as indicated,



$$\frac{x}{r} = \cos\theta$$

$$\frac{y}{r} = \sin\theta$$

$$r = |z| = \sqrt{x^2+y^2}$$

$$\therefore z = x+iy = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$$

- θ is not uniquely determined ; $\theta = \arg(z)$, argument of z
  ($\theta' = \theta + 2n\pi$, $n \in \mathbb{Z}$ also works)
- if we assume in addition that $-\pi < \theta \leq \pi$,

$\theta = \operatorname{Arg}(z)$ Principal argument of z

- Euler 1748 $e^z = 1 + z + \frac{z^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

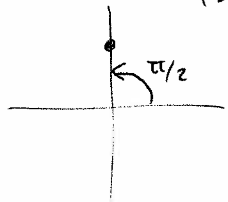
$$e^{i\theta} \stackrel{!}{=} \cos\theta + i\sin\theta \quad \text{(compare power series....)}$$

$$\therefore z = re^{i\theta} \longleftarrow \text{Polar form of cplx \# } z$$

- $re^{i\theta} = r'e^{i\theta'} \iff r = r',\ \theta - \theta' = 2n\pi,\ n \in \mathbb{Z}$

Polar form examples

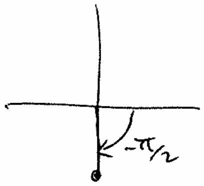
e.g. (a) i $\qquad |i| = \sqrt{0^2+1^2} = 1$ $\qquad$ (R4)



$$i = 1 \cdot e^{i\pi/2} = (\cos \pi/2 + i \sin \pi/2)$$

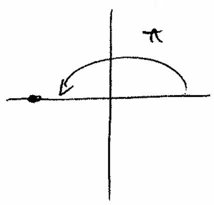
w04 ↓ Jan 6
Jan 8

(b) $-i$



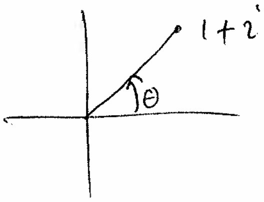
$$-i = e^{-i\pi/2} \qquad (-i = e^{i3\pi/2} \ \ldots)$$

(c) -1



$$|-1| = 1 \qquad -1 = e^{i\pi} \qquad (= e^{-i\pi}, e^{3i\pi}, \ldots$$

(d)

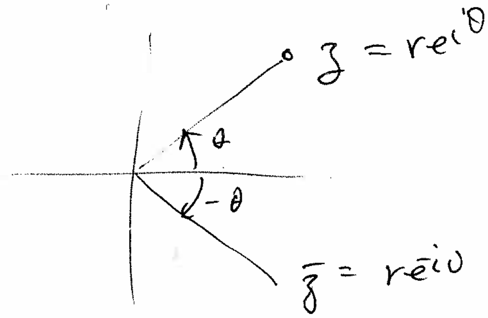


$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\left.\begin{array}{l} \cos\theta = \frac{1}{\sqrt{2}} \\ \sin\theta = \frac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow \theta = \pi/4$$

$$\therefore \quad 1+i = \frac{\sqrt{2}}{2} e^{i\pi/4}$$

Rmk. $\overline{(re^{i\theta})} = re^{-i\theta}$



$|e^{i\theta}| = 1$ for all θ.

$x \neq 0 \quad \frac{y}{x} = \tan\theta$ not enough

<u>multⁿ complex #'s</u> : (easy in polar form)

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2};$$

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2)\, e^{i\theta_1} \cdot e^{i\theta_2}$$

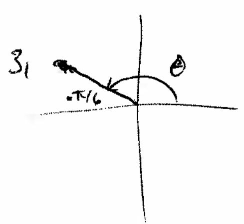
$$= (r_1 r_2)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = (r_1 r_2)\big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\big]$$

$$= r_1 r_2\big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big] = r_1 r_2\, e^{i(\theta_1+\theta_2)} \quad !$$

Similarly, if $z_2 \neq 0$, $\underline{\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}}$.

e.g. polar form of $z = \dfrac{-\sqrt{3}+i}{1+i} = \dfrac{z_1}{z_2}$;

<u>$-\sqrt{3}+i$</u>; $|-\sqrt{3}+i| = \sqrt{3+1} = 2$; $|1+i| = \sqrt{2}$; $1+i = \sqrt{2}e^{i\pi/4}$



$$\left.\begin{aligned}\cos\theta &= -\tfrac{\sqrt{3}}{2}\\ \sin\theta &= \tfrac{1}{2}\end{aligned}\right\} \Rightarrow \theta = 5\pi/6$$

$$z_1 = 2e^{i5\pi/6}$$

$$\therefore\ z = \frac{2e^{i5\pi/6}}{\sqrt{2}\,e^{i\pi/4}} = \sqrt{2}\, e^{i\pi(5/6 - 1/4)} = \sqrt{2}\, e^{i\frac{7\pi}{12}}$$

Remarks: <u>Gauss (1797)</u> If $n \geq 1$, Every polynomial $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with complex coefficients can be completely factored using complex #'s, i.e.

$$p(z) = a_n(z-w_1)(z-w_2)\cdots(z-w_n) \quad \text{with } a_n, w_i \in \mathbb{C}$$

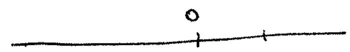
roots

Vectors in $\mathbb{R}^n$ Vector (Latin: vehere: to convey, carrier") (R6)

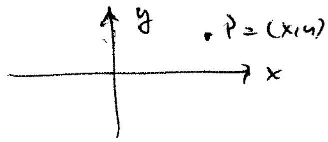
We'll use intuition from low dimensions for higher dimensions (R7)

geometric insights + algebraic understanding ⟶ "geometric ideas" algebraic ✓

"Alg"   "Geom"



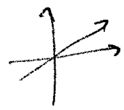
$\mathbb{R}$ - scalars



Cartesian plane

(17) $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$; $u = (x,y)$

$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$ $v = (x,y,z)$



'3-space'

⋮ 300 years

space-time?

Hamilton $\mathbb{R}^4 = \{(x_1,x_2,x_3,x_4) \mid x_i \in \mathbb{R}\}$

? "n-space" (4.1.2)

$n \in \mathbb{N}$ $\mathbb{R}^n = \{(x_1,\ldots,x_n) \mid x_i \in \mathbb{R}\}$; $u = (x_1,\ldots,x_n)$

↑ ellipsis

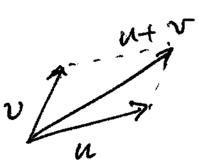
Manipulations of vectors in $\mathbb{R}^n$: Eq $(x_1,\ldots,x_n) = (x_1',\ldots,x_n') \Leftrightarrow x_i = x_i'$, $i = 1,\ldots,n$.

Add^n $(x_1,\ldots,x_n) + (x_1',\ldots,x_n') = (x_1+x_1',\ldots,x_n+x_n')$

Zero vector $(0,\ldots,0) \in \mathbb{R}^n$,   Negative $-(x_1,\ldots,x_n) = (-x_1,\ldots,-x_n)$

Mult^n by scalar $k \in \mathbb{R}$, $k(x_1,\ldots,x_n) = (kx_1,\ldots,kx_n)$. (

$n = 2$.



$k\,(>0)$

$k'\,(<0)$

① length ku $= |k|$. length u

② same dir^n.

Def^n If $k_1,\ldots,k_m \in \mathbb{R}$, and $u_1,\ldots,u_m \in \mathbb{R}^n$ are vectors,
$k_1u_1 + k_2u_2 + \cdots + k_mu_m$ is called a l.c. of $u_1,\ldots,u_m$

N.B. All usual properties hold: $\forall u,v,w \in \mathbb{R}^n$, $\forall k \in \mathbb{R}$

$(u+v)+w = u+(v+w)$	$u+v = v+u$	$k(k'u) = (kk')u$
$u+0 = u$	$k(u+v) = ku+kv$	$1u = u$.
$u+(-u) = 0$	$(k+k')u = ku+k'u$	

L2/L3

3.2 + Dot (Inner Product) in $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$: (length + angles!) (R8)

$\underline{n=3}$ $u = (x_1, x_2, x_3)$ $v = (y_1, y_2, y_3)$ vecs. in $\mathbb{R}^3$

$u \cdot v = x_1y_1 + x_2y_2 + x_3y_3$ <u>dot product</u>

$\|u\| = \sqrt{x_1^2+x_2^2+x_3^2}$ <u>length</u>, <u>norm</u> of u

$\|u-v\|$ = distance between u & v

$\underline{n}$ $u = (x_1, \dots, x_n)$, $v = (y_1, \dots, y_n)$; $u \cdot v = x_1y_1 + \dots + x_ny_n$

$\|u\| = \sqrt{x_1^2 + \dots + x_n^2}$

$\mathbb{R}^4$ with "$\cdot$" <u>"Euclidean n-space"</u>

<u>N.B.</u> : $u \in \mathbb{R}^n$ is zero $\iff \|u\| = 0$.

L2/L3 $\boxed{}$

N.B. $n \le$ For $n \le 3$, seen $u, v \neq 0$ $u \cdot v = 0 \iff u$ and v are <u>perpendicular</u> <u>orthogonal</u>

$\underline{n}$; u, $u \cdot v = 0$ say u, v are perpendicular.

e.g. $(1,0,0,1), (1,0,0,-1)$ are $\perp$, since $(1,0,0,1)\cdot(1,0,0,-1) = 0$

(340!)

<u>other angles</u>?

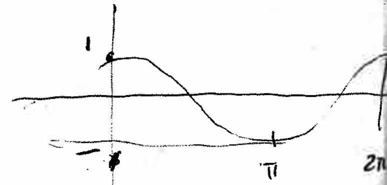
L2/L3

<u>Prob 26</u>, <u>Ex 3.2</u> (Cauchy–Schwarz) If $u, v \in \mathbb{R}^n$, then

$|(u \cdot v)| \le \|u\|\,\|v\|$; $\Rightarrow$ ① $\|u+v\| \le \|u\| + \|v\|$ ($\triangle$ inequality) (like for angle #s).

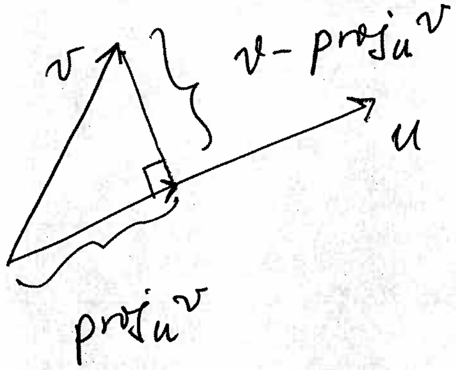
$\Rightarrow$ <u>Allows</u> defn of angle between 2 non-zero vecs. If $u, v \neq 0$, angle θ between u & v is θ which satisfies

1) $\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$ $(|\;| \le 1)$



2) $0 \le \theta \le \pi$ (so it's unique $\le \pi$.)

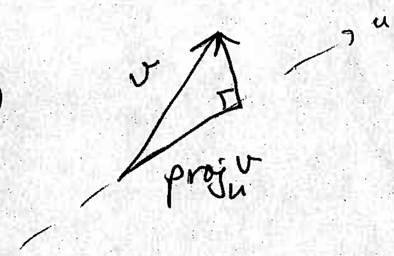
Orthogonal Projections of v on u 3.2.3   R9



$$proj_u v = \underbrace{\frac{(v \cdot u)}{\|u\|^2}}_{\text{scalar}} \underbrace{u}_{\text{vector}}$$

Satisfies 1) $proj_u v \parallel u$ ($\Rightarrow$ formula above)

2) $v - proj_u v \perp u$



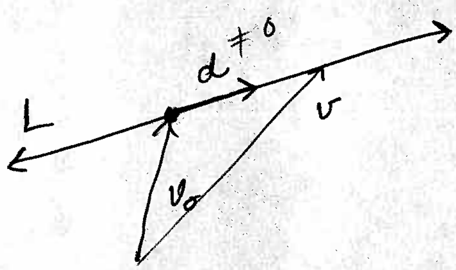
So $v = \underset{\parallel v}{proj_u v} + \underset{\perp}{(v - proj_u v)}$

transmission
UMTS System

Lines in $\mathbb{R}^2, \mathbb{R}^3$ $(\mathbb{R}^n)$

$L = \{ v_0 + t\underset{\uparrow}{d} \mid t \in \mathbb{R}, d \neq 0, d \in \mathbb{R}^n \text{ fixed} \}$

dir^n vector of line



$v = v_0 + td$

"Line through v_0 with dir^n d"

- vector form above

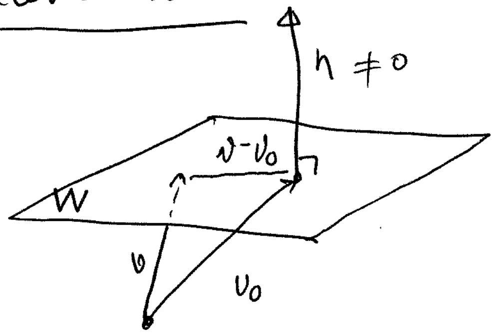
- write $v = (x_1, x_2, x_3)$ $d = (d_1, d_2, d_3)$

$(n=3)$ $v_0 = (a, b, c)$

$x_1 = a + td_1$
$x_2 = b + td_2$
$x_3 = c + td_3$

R10

Planes in $\mathbb{R}^3$



$$W = \{ v \in \mathbb{R}^3 \mid (v - v_0) \cdot n = 0 \}$$

Plane through v_0 with normal n.

(non-zero multiples ok)

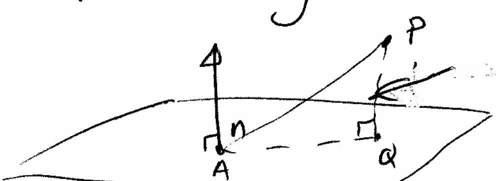
Cartesian form: eg $n = (2, -1, 1)$, $v_0 = (0, 0, 3)$

$$W = \{ (x, y, z) \in \mathbb{R}^3 \mid (x, y, z - 3) \cdot (2, -1, 1) = 0 \}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 \mid 2x - y + z = 3 \}$$

coeff^n $\Rightarrow$ normal.

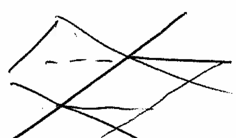
– intersection of lines, lines & planes, planes & planes



shortest distance of P from plane is

e.g. $\|P - Q\| = \| \operatorname{proj}_n (P - A) \|$

– "



"angle" between planes $\equiv$ angle between normals.

Cross Product in $\mathbb{R}^3$

3.3.4, 3.5 ($\sim$ area, volume)

$u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \left(y_1 z_2 - y_2 z_1, -(x_1 z_2 - x_2 z_1), x_1 y_2 - x_2 y_1 \right)$$

$$= \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)$$

Thms 2, 3

176 – 177

$$(u \times v) \cdot u = 0 = (u \times v) \cdot v$$

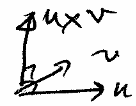
$$u \times v = - v \times u$$

distrib law ✓

Rmk. $u \times (v \times w) \neq (u \times v) \times w$ (not associative) (R4)

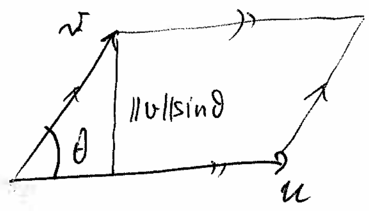
eg $\hat{\imath} \times (\vec{k} \times \vec{k}) = \hat{\imath} \times 0 = 0$

$(\hat{\imath} \times \hat{k}) \times \vec{k} = -\hat{\jmath} \times \vec{k} = -\hat{\imath}$ $\neq$

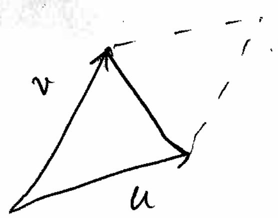
Thm 5 / 178 . $\|u \times v\| = \|u\| \, \|v\| \sin\theta$  RH-rule

$\Rightarrow$ ① u, v "$\parallel$" $\Longleftrightarrow$ $u \times v = 0$

(later: (ex.) $u \times v = 0 \Rightarrow$ u is a multiple of v or v " u.).

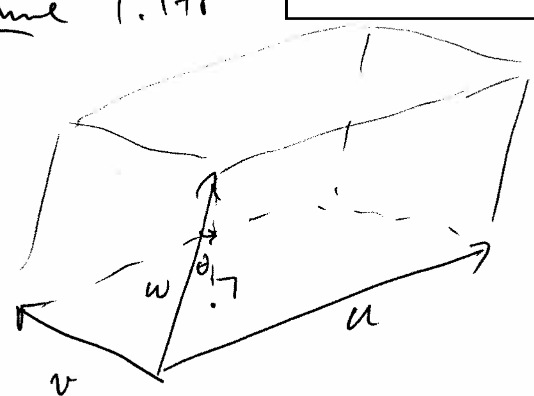
$\Rightarrow$ ②  area $\|$ogram $= \|u\| \, \|v\| \sin\theta$

$= \|u \times v\|$

 area $\triangle = \frac{1}{2} \|u \times v\|$.

Volume P. 178

$u \times v \neq 0$
$w \neq 0$



vol = area base . $\perp$ ht

$= \|u \times v\| \cdot \perp$ ht

$\perp$ ht $= \|w\| |\cos\theta|$, θ as shown

But $|\cos\theta| = \left| \dfrac{(u \times v) \cdot w}{\|u \times v\| \, \|w\|} \right|$ so

Thm 6 P. 178 (ignore "det") vol $= \dfrac{\cancel{\|u \times v\|} \cdot \cancel{\|w\|} \cdot |(u \times v) \cdot w|}{\cancel{\|u \times v\|} \; \cancel{\|w\|}} = |u \times v \cdot w|$

R12

e.g. find vol llepiped det. by $\overset{u}{(1,1,1)}$, $\underset{v}{(1,2,7)}$ & $\underset{w}{(2,3,6)}$. (order isn't imp.)

$$v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 7 \\ 2 & 3 & 6 \end{vmatrix} = (-9, -(-8), -1)$$

$$u \cdot (v \times w) = (1,1,1) \cdot (-9, 8, -1) = -2$$

$\therefore$ vol $= +2$.

L3/L4

Remarks

- $(u \times v) \cdot w = u \cdot (v \times w)$
$= w \cdot u \times v$
$= w \times u \cdot v$ etc
(same cyclic order)

- 3 vectors are coplanar (can be made to lie in same plane) $\iff u \cdot v \times w = 0$
later $\iff$ there are scalars a, b, c not all zero st. $au + bv + cw = 0$.

- $u \cdot v \times w = \begin{vmatrix} u \\ v \\ w \end{vmatrix}$ 3×3 determinant (later) e.g. above $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 7 \\ 2 & 3 & 6 \end{vmatrix}$

L_3/L_4

Chapters 4,5 Vector Spaces

(examples from 1.1)

1

$\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ – geometric vectors

$\mathbb{R}^n$, $n \geq 4$ "vectors"

NON-geometric, NON-"$\mathbb{R}^n$" vectors ...

e.g. "Spaces" of equations.

E_1: $x - y - z = 1$
E_2: $2x - y + z = 1$
E_3: $-x + 2y + 4z = 4$

$E_2 - 2E_1 = E_4 : y + 3z = 3$

$E_1 + E_3$: $y + 3z = 3$

; E_5 : $E_1 + E_4$: $x + 2z = 2$

We can write : $E_1 + E_3 = E_2 - 2E_1$
$3E_1 - E_2 + E_3 = 0$? ...

Remarks: 1) can add eqns to get another eqn

2) can multiply " by a scalar to get another eqn

3) equations have negatives!

$-E_1$: $-x - y + z = 1$

$E_1 + (-E_1) = ?$ "zero equation"

4) "zero equation" $0 = 0$ works!

5) Certain usual arithmetic holds:

$E_1 + E_2 = E_2 + E_1$

$E_1 + (E_2 + E_3) = (E_1 + E_2) + E_3$

$k(E_1 + E_2) = kE_1 + kE_2$ $k \in \mathbb{R}$

⋮

These are exactly the (10) properties of $\mathbb{R}^n$ (Thm 1.1., P.4) !

One could consider the space $\mathcal{E}$ of all equations "obtainable from" generated by E_1 & E_2 & E_3 Note that

$\mathcal{E}$ behaves exactly like a space of vectors! "linear combinations of equations"

$\mathcal{E} = \{k_1E_1 + k_2E + k_3E_3 \mid k_i \in \mathbb{R}\}$!

(Q: is there an eq of the form $y = y_0$ $x = x_0$? in $\mathcal{E}$?

i.e. can we solve for y?

i.e. Can we find y_0, a, $b \in \mathbb{R}$ s.t.

$(y = y_0)$ is $aE_1 + bE_2$?

exercise: no

- e.g. Formation of HBr (Stoichiometry)

— see web notes P1, 2.

(H

"vectors" don't have to be geometric, or even "n-tuples"

1.1 What do we really need here?

V - set of vectors of some sort; operations: "add^n" of vectors, "mult^n" of vectors by scalar

"Closure"
- the sum of 2 vectors should be a vector; $u, v \in V \Rightarrow u+v \in V$
- the scalar multiple of a vector should be a vector $a \in \mathbb{R},\ v \in V \Rightarrow a v \in V$

"Existence"
- there should be a zero vector 0; $0 + v = v$
- every vector should have a negative
  given $v \in V$, there is $-v \in V$ s.t.
  $v + (-v) = 0$ (previous
  $-v = -1 . v$?

- Arithmetic properties:

$$v + w = w + v$$
$$u + (v + w) = (u + v) + w$$
$$a(v + w) = av + aw$$
$$(a + b)v = av + bv$$
$$a(bv) = (ab)v$$
$$1 . v = v$$

Any set V, with 2 operations as above satisfying these (10!) these conditions "axioms" is called a <u>VECTOR SPACE</u>

(See P293, (5.1))

(4)

e.g. $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{C}$ with usual ops.

eg $V = \{(x, 2x) \mid x \in \mathbb{R}\}$, standard ops from $\mathbb{R}^2$

C ✓ E ✓ A — ok, all works for $\mathbb{R}^2$ ✓

eg $U = \{(x, x+2) \mid x \in \mathbb{R}\}$; usual ops of $\mathbb{R}^2$

Usual ops $\Rightarrow$ usual zero $= (0,0) \notin U$. $\therefore$ ✗

eg $M_{22}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$

Add^n, mult^n by scalars : componentwise

C ✓ E ✓ A (like $\mathbb{R}^4$, stacked).

($M_{mn}(\mathbb{R}) = \{ A \mid A$ is an $m \times n$ matrix with real entries$\}$)

eg <u>Spaces of functions</u> (#5, P.295)

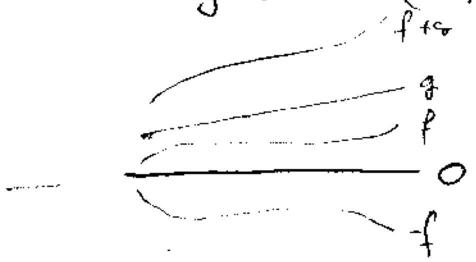
$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$

$F[a,b] = \{ f \mid f : [a,b] \to \mathbb{R} \}$ N.B. $f = g \Leftrightarrow f(x) = g(x)$ $\forall x \in [a,b]$

$f, g \in F[a,b]$; $k \in \mathbb{R}$

<u>Add</u>^{n} $(f+g)(x) = f(x) + g(x)$; $f+g \in F[a,b]$

mult^n by scalar $(kf)(x) = k.f(x)$; $kf \in F[a,b]$



C ✓

E $0(x) = 0$, $\forall x \in [a,b]$

$(-f)(x) = -f(x)$

A ✓

- Can also consider

$F(\mathbb{R}) = \{f \mid f: \mathbb{R} \to \mathbb{R}\}$, same ops ("ptwise")

$f(x) = \cos x$, $f \in F(\mathbb{R})$

$g(x) = x + x^2$, $g \in F(\mathbb{R})$, all polynomial function

$h(x) = \frac{1}{x}$

$t(x) = \tan x$ $\Big\} \notin F(\mathbb{R})$; not def'd everywhere.

5.1.2 Subspaces & Spanning Sets

Suppose V is a v.s. and $W \subset V$ is a subset of V

(eg $\{(x, 2x) \mid x \in \mathbb{R}\}$). Give W same ops as V.

Is W a v.s.?

C – add^n ?
– mult by scalars ?

E – zero ?
– -ves $-1 \cdot v = -v$ ...
(true from axioms! Thm 2, p 296)

A – true in V, so true in W

Def^n A subset W of a v.s V is a subspace of V if W is a v.s. when given the same ops as V.

e.g. $\{(x, 2x) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$ | e.g. $\{0\}$ is a s.s of V; V " " of V

Thm 3 (Subspace Test) If V is a v.s. and $W \subset V$, then
P. 297 W is a subspace of V iff

SHORTCUT
1. $0 \in W$
2. W is closed under add^n
3. " mult^n by scalars.

eg $T = \{ v \in \mathbb{R}^3 \mid v \cdot (1,-2,1) = 0 \} \subset \mathbb{R}^3$
$= \{ (x,y,z) \mid x - 2y + z = 0 \}$ $(\neq \mathbb{R}^3)$

<u>Subspace Test</u>

1) $\underset{(0,0,0)}{0} \in T$ since $(0,0,0) \cdot (1,-2,1) = 0$

2) If $u, v \in T$, then

$$(u+v) \cdot (1,-2,1) = u \cdot (1,-2,1) + v(1,-2,1)$$
$$= 0 + 0$$
$$= 0$$

$\therefore u+v \in T$ $\quad$ $\therefore T$ closed under $+$

3) If $u \in T$, $k \in \mathbb{R}$, then

$$(ku) \cdot (1,-2,1) = k. (u \cdot (1,-2,1))$$
$$= k . 0$$
$$= 0$$

$\therefore ku \in T.$ $\quad$ $\therefore T$ closed under scalar mult by s.

$\therefore T$ is a subspace of $\mathbb{R}^3$

<u>Remarks</u> ① <u>Any</u> plane through origin is a subspace of $\mathbb{R}^3$ - there's nothing special about $(1,-2,1) = n$.

② If a plane in $\mathbb{R}^3$ doesn't contain 0, it's <u>not</u> a subspace of $\mathbb{R}^3$

③ Any line in $\mathbb{R}^3$ (or $\mathbb{R}^2$) thru 0 is a s.s. of $\mathbb{R}^3$ (or $\mathbb{R}^2$)
(Same remark about lines not containing 0)

e.g. $V = F[0,1]$, $W = \{ f \in F[0,1] \mid f(0) = 0 \}$

0 : $0(0) = 0$ so $0 \in W$

$+$: $f, g \in W \Rightarrow (f+g)(0) = f(0) + g(0) = 0$ $\therefore f+g \in W$

k : $f \in W, k \in \mathbb{R} \Rightarrow (kf)(0) = k.f(0) = k.0 = 0$ $\therefore kf \in W$

$\therefore W$ is a subspace of $F[0,1]$

Transpose (P.7) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t := \begin{bmatrix} a & c \\ b & d \end{bmatrix}$; can do it for $m \times n$ matrices (7)

e.g. $S = \{ A \in M_{22} \mid A^t = A \}$ Symmetric 2x2 matrices

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}.$

0: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore$ $0 \in S$

$+$: $\begin{bmatrix} a & b \\ b & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ b' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ b+b' & d+d' \end{bmatrix} \in S$ $\therefore$ S closed under $+$

$k\cdot$: $k \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kb & kd \end{bmatrix} \in S$ $\therefore$ S closed under mult. by scalars.

$\therefore$ S is a subspace of M_{22}

e.g. trace of square matrix (can do it for $n \times n$'s)

trace $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$;

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \operatorname{tr} A = a + d$ see #.

$SL_2 = \{ A \in M_{22} \mid \operatorname{tr} A = 0 \}$ is a subspace of M_{22}

$\left(= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} \right)$ . . .

Infinite $\rightsquigarrow$ finite

Spans & linear combinations (P. 299)

Recall

$T = \{(x,y,z) \mid x-2y+z=0\}$ SS of $\mathbb{R}^3$

$S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a,b,d \in \mathbb{R} \right\}$ SS of M_{22}

Rewrite T: $T = \{(x,y,z) \mid x = 2y - z\}$

$= \{(2y-z, y, z) \mid y, z \in \mathbb{R}\}$ no conditions left

$= \{(2y, y, 0) + (-z, 0, z) \mid y, z \in \mathbb{R}\}$

$= \{\underbrace{y(2,1,0) + z(-1,0,1)}_{} \mid y, z \in \mathbb{R}\}$

linear comb of $(2,1,0)$ & $(-1,0,1)$

" $= \text{span}\{(2,1,0), (-1,0,1)\}$ "

$S = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a,b,d \in \mathbb{R} \right\}$

" $= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ "

Def^{n}s (P. 299) ① If $v_1, \dots, v_m$ are vectors in a v.s. V, and $a_1, \dots, a_m$ are scalars, the vector

$a_1 v_1 + \cdots + a_m v_m$ is called a

linear combination of $v_1, \dots, v_m$.

② $\{a_1 v_1 + \cdots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{R}\}$ is called the span of $v_1, \dots, v_m$; we write

$\text{span}\{v_1, \dots, v_m\} = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{R}\}$

$\{v_1, \dots, v_m\}$ is called a spanning set for $\text{span}\{v_1, \dots, v_m\}$

OR " spans "

(3) A vector space (or subspace) W is spanned by $v_1, \dots, v_m \in W$ if $W = \text{span}\{v_1, \dots, v_m\}$; "$\{v_1, \dots, v_m\}$ spans W"

e.g. $T = \{ y(2,1,0) + z(-1,0,1) \mid y, z \in \mathbb{R} \}$

$= \text{Span}\{ \underset{v_1}{(2,1,0)}, \underset{v_2}{(-1,0,1)} \}$; $\{v_1, v_2\}$ spans T

$\{v_1, v_2\}$ is a spanning set for T

$S = \text{Span}\left\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{M_3}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$

$\{M_1, M_2, M_3\}$ spans S; S is spanned by M_1, M_2 & M_3.

e.g. Let $L = \text{Span}\{(0,1,0)\} = \{ y(0,1,0) \mid a \in \mathbb{R} \}$
$= \{ (0, y, 0) \mid y \in \mathbb{R} \}$ (y-axis)

is a s.s. of $\mathbb{R}^3$! (like others...)

<u>Thm 4 P. 299</u> Let V be a v.s. If $\{v_1, \dots, v_m\} \subset V$, then

(1) $U = \text{Span}\{v_1, \dots, v_m\}$ <u>is always</u> a subspace of V

(2) If W is <u>any</u> s.s. of V s.t. $W \supseteq \{v_1, \dots, v_m\}$ then
$$W \supseteq U$$

i.e. $U = \text{Span}\{v_1, \dots, v_m\}$ is the <u>smallest s.s</u> of V containing $v_1, \dots, v_m$.

Pf. (1) (Rem s.s. test.) I $0 = 0v_1 + 0v_2 + \dots + 0v_m \in U$

II If $u = a_1v_1 + \dots + a_mv_m$ & $v = b_1v_1 + \dots + b_mv_m$, then
$u + v = (a_1 + b_1)v_1 + \dots + (a_m + b_m)v_m \in U$

III If $u = a_1v_1 + \dots + a_mv_m$ & $k \in \mathbb{R}$,
$ku = (ka_1)v_1 + \dots + (ka_m)v_m \in U$.

(2) - exercise.

e.g. $\{(x,y,x-y) \mid x,y \in \mathbb{R}\} = \text{span}\{(1,0,1),(0,1,-1)\}$
$\therefore$ is a S.S. of $\mathbb{R}^3$

e.g. $V = F(\mathbb{R})$, $H = \text{span}\{f,g\}$ is a S.S. of $F(\mathbb{R})$!
$f(x) = \cos x$ $= \{af + bg \mid a,b \in \mathbb{R}\}$
$g(x) = \sin x$ (ex. show, if $h(x) = \sin(x+1)$, $= (\sin x)(\cos 1) + (\sin 1)\cos x$
then $h \in H$!
hint: trig identity for $\sin(a+b)$)
$= a \sin x + b \cos x$
$(a = \cos 1,\ b = \sin 1)$

note: if $k(x) = 1$, $\forall x \in \mathbb{R}$, then $k \notin H$:
$k = af + bg$, fixed $a, b \in \mathbb{R}$
Suppose $k(x) = a\cos x + b \sin x$, $\forall x \in \mathbb{R}$

Choose convenient values of x :

$x = 0$: $k(0) = 1 = a.\cos 0 + b.\sin 0$
$\Rightarrow 1 = a$

$x = \frac{\pi}{2}$: $k(\frac{\pi}{2}) = 1 = a\cos \pi/2 + b \sin \pi/2$
$\Rightarrow 1 = b$

$x = \pi$: $k(\pi) = 1 = a\cos\pi + b\sin\pi$
$\Rightarrow 1 = -a$

C!

$\therefore k \notin \text{span}\{f,g\} = H$.

e.g.

9a.

e.g. 4.1(c) $V = F(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$ v.s.

$W = \{ f \in F(\mathbb{R}) \mid f(-x) = -f(x) \ \forall x \in \mathbb{R} \}$ "odd" f^{ns}.

1. $0 \in W$ since $0(-x) = 0 = -0 = -0(x), \ \forall x \in \mathbb{R}$

2., 3. If $f, g \in W$ and $k \in \mathbb{R}$

$k=1$ add^n
$f=0$ mult^n scalars

$$(f + kg)(-x) = f(-x) + k\,g(-x) = -f(x) - kg(x)$$
$$= -(f(x) + kg(x))$$
$$= -(f + kg)(x)$$

$\therefore f + kg \in W$.

eg V any vector space

$\{0\} \subseteq V$ is a s.s. of V

$V \subseteq V$ is a s.s. of V

e.g. Diagonal matrices (2×2)

$D_2 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$ is a SS of M_{22}

because $D_2 = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$D_3 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

$= \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$ is a

S.S. of M_{33}

e.g. <u>Subspaces of $\mathbb{R}$</u>

$\{0\}$ ✓

any others?

$\mathbb{R}$ ✓

Suppose U is a S.S. of $\mathbb{R}$, and that $U \neq \{0\}$; i.e. there is $u \neq 0$, $u \in U$. Claim: $U = \mathbb{R}$. Let $r \in \mathbb{R}$ be any real number.

e.g. <u>Subspaces of $\mathbb{R}^2$</u> $\quad \{(0,0)\}$, $\mathbb{R}^2$, lines thru $(0,0)$. (14)

<u>Theorem</u>: these <u>are</u> the only s.s. of $\mathbb{R}^2$.

Suppose W is a s.s. of $\mathbb{R}^2$ and $W \neq \{(0,0)\}$ i.e. W contains a non-zero vector w_0 ($\therefore$ a lot many!), so $\text{span}\{w_0\} \subset W$

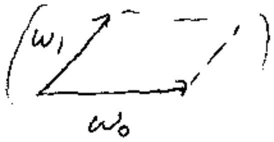
<u>Case 1</u> $\quad W = \text{span}\{w_0\}$. In this case W is the line through 0 with dir^n w_0.

<u>Case 2</u> There is a vector in W which is not in $\text{span}\{w_0\}$ i.e. there is $0 \neq w_1 \in W$ s.t. w_1 and w_0 are <u>not</u> parallel.

Then, $\text{span}\{w_0, w_1\} \overset{\text{as } W \text{ is a s.s.}}{\subset} W$; <u>claim</u>: $\text{span}\{w_0, w_1\} = \mathbb{R}^2 = W$.

Since $\text{span}\{w_0, w_1\} \subset W \subset \mathbb{R}^2$, it's enough to show that $\mathbb{R}^2 \subseteq \text{span}\{w_1, w_2\}$ i.e. any $v \in \mathbb{R}^2$ is a l.c. of w_0 & w_1

The fact that w_0, w_1 aren't $\parallel$ means that

area () $\neq 0$ i.e. if $w_0 = (x_0, y_0)$, $w_1 = (x_1, y_1)$ Then

$$x_0 y_1 - x_1 y_0 \neq 0.$$

Then, if $(x, y) \in \mathbb{R}^2$, one can easily check that

$$(x, y) = \left(\frac{x y_1 - y x_1}{x_0 y_1 - x_1 y_0}\right) w_0 + \left(\frac{x_0 y - y_0 x}{x_0 y_1 - x_1 y_0}\right) w_1$$

(check x comp: $\dfrac{(x y_1 - y x_1) x_0 + (x_0 y - y_0 x) x_1}{x_0 y_1 - x_1 y_0} = \dfrac{x\,(x_0 y_1 - x_1 y_0)}{''} = x$.

nonzero vectors

e.g. Subspaces of $\mathbb{R}^3$ $\quad \{(0,0,0)\}$, $\mathbb{R}^3$, lines thru 0, planes thru 0.

Theorem (later) These are the only subspaces of $\mathbb{R}^3$.

for now: if $V = \operatorname{span}\{v_0\}$, $v_0 \neq 0$ in $\mathbb{R}^3$, V is a line

if $V = \operatorname{span}\{v_0, v_1\}$ and nonzero vs v_0, v_1 aren't parallel,

then: claim $V = \{v \in \mathbb{R}^3 \mid v \cdot (v_0 \times v_1) = 0\}$ is the plane through 0 with normal $v_0 \times v_1$!

(i) $V \subseteq$ plane described above: let $v = a v_0 + b v_1$. Then

$$v \cdot (v_0 \times v_1) = (a v_0 + b v_1) \cdot (v_0 \times v_1) = a\,(v_0 \cdot v_0 \times v_1) + b\,(v_1 \cdot v_0 \times v_1)$$
$$= a \cdot 0 + b \cdot 0$$
$$= 0$$

$\therefore$ $v \in$ plane.

(ii) plane above $\subseteq V$ i.e. $v \cdot (v_0 \times v_1) = 0 \implies v \in \operatorname{span}\{v_0, v_1\}$

Let $n = v_0 \times v_1$. Then $n \neq 0$ because v_0, v_1 aren't parallel.

Recall $u_1 \times (u_2 \times u_3) = (u_1 \cdot u_3)\, u_2 - (u_1 \cdot u_2)\, u_3$ $\quad *$ $\quad$ (1.65(f))

Suppose $v \cdot n = 0$.

Then $n \times (v \times n) = \|n\|^2 v - \cancel{(n \cdot v)}\, n = \|n\|^2 v$ $\quad$ [illegible] $\times$ [illegible] $=$ [illegible]

Now expand $n \times (\underbrace{v \times n})$ using get

$v \times n = a v_0 + b v_1$

$\boxed{\begin{aligned} &\operatorname{span}\{(1,[illegible]),(0,[illegible])\} \\ &= \{v \mid v \cdot (-1,-1,[illegible]) = 0\} \\ &= \{(x,y,z) \mid [illegible] = 0\} \\ &= \operatorname{span}\{(1,1,2), (-1,1,0)\} \end{aligned}}$

$= n \times (a v_0 + b v_1)$

$= -\,a\, v_0 \times \underbrace{n}_{v_0 \times v_1} - b\, v_1 \times \underbrace{n}_{v_0 \times v_1}$ $\quad$ expand again

$= a' v_0 + b' v_1$ $\quad$ See web page for a'

4.7 Linear Dependence & Independence

generalize geometric notions

2 vectors are / aren't parallel, lie / don't lie in a line — collinear

3 vectors lie / don't lie in a plane
are / are not coplanar

need to express these ideas algebraically
– necessitates talking about all vectors, including 0, include degenerate ca

Suppose 2 vectors u, v are collinear
This is equivalent to: either $u = kv$ for some $k \in \mathbb{R}$ (unless $v=0$, $u \neq 0$)
or $v = lu$ for some $l \in \mathbb{R}$ (…)

These two statements can be summarized as follows:
There are scalars $a, b \in \mathbb{R}$, not both zero s.t. $au + bv = 0$
($a \neq 0 \Rightarrow u = (-\frac{b}{a})v$; or $b \neq 0 \Rightarrow v = (-\frac{a}{b})u$)

Similarly, "3 vectors u, v, w lie in a plane" is equivalent to
there are scalars $a, b, c \in \mathbb{R}$, not all zero, s.t.
$$au + bv + cw = 0.$$

CONTRAPOSITIVE is also important: 2 vectors u, v do not lie in a line ⟸

2 non zero vectors u, v are not parallel $\Longleftrightarrow$
$$au + bv = 0 \implies a = b = 0$$
(⟸ clear true, but has no informa…)

3 vectors u, v, w do not lie in a plane $\Longleftrightarrow$ for scalars a, b, c,
$$au + bv + cw = 0 \implies a = b = c = 0$$

If V is a vector space, $v_1, \dots, v_m$ are vectors in V 17

Def^n P.126 $\{v_1, \dots, v_m\}$ is / $v_1, \dots, v_m$ are <u>linearly dependent</u>

if there are scalars $a_1, \dots, a_m$, not <u>all</u> zero, s.t.

$$a_1 v_1 + \dots + a_m v_m = 0.$$

$\{v_1, \dots, v_m\}$ is / $v_1, \dots, v_m$ are <u>linearly independent</u>

if for scalars $a_1, \dots, a_m$, $a_1 v_1 + \dots + a_m v_m = 0 \underset{\text{implies}}{\Longrightarrow} a_1 = a_2 = \dots = a_m = 0$

$$\boxed{\text{l.i.} = \neg(\text{l.d.})}$$

---

e.g. $\{(1,0),(0,1)\}$ is l.i. since $a_1(1,0) + a_2(0,1) = (0,0)$
$\Rightarrow (a_1, a_2) = (0,0) \Rightarrow a_1 = a_2 = 0$

$\{(1,-1),(1,1)\}$ is l.i. since $a_1(1,-1) + a_2(1,1) = (0,0)$

$$\Rightarrow \left.\begin{array}{l} a_1 + a_2 = 0 \\ -a_1 + a_2 = 0 \end{array}\right\} \Rightarrow a_1 = a_2 = 0$$

- $\{(1,0),(0,1),(1,1)\}$ is <u>l.d.</u> since $1\cdot(1,0) + 1\cdot(0,1) - 1\cdot(1,1) = (0,0)$!
(e.g.)

<u>rmk</u>: $(1,0) \in \operatorname{span}\{(0,1),(1,1)\}$

$\Big\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{M_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{M_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \Big\}$ is l.i. since

$$a_1 M_1 + a_2 M_2 + a_3 M_3 + a_4 M_4 = 0 \Rightarrow \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$$
$$a_1 = \dots = a_4 = 0.$$

$\Big\{ \underset{A_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{A_2}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{A_3}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \Big\}$ is l.d. since $A_1 - A_2 + A_3 = 0$

rmk: $A_2 \in \operatorname{span}\{A_1, A_3\}$

e.g. $\{1, \sin x, \cos x\}$ is l.i. in $F[0, 2\pi]$:

Suppose $a1 + b\sin x + c.\cos x = 0$ $\underline{\forall x \in [0, 2\pi]}$. $*$

$$\left.\begin{array}{lll} \text{at } x = 0 & * \Rightarrow & a + c = 0 \\ x = \pi/2 & * \Rightarrow & a + b = 0 \\ x = \pi & * \rightarrow & a - c = 0 \end{array}\right\} \Rightarrow a = b = c = 0.$$

e.g. $\{\overset{f}{1}, \overset{g}{\sin^2 x}, \overset{h}{\cos^2 x}\}$ is l.d. in $F[0, 2\pi]$

Since $1 - \sin^2 x - \cos^2 x = 0$, $\underline{\forall x} \in [0, 2\pi]$.

i.e. $f - g - h = 0$ as functions.

rmk: $\cos^2 x \in \text{span}\{1, \sin^2 x\}$

$\Rightarrow \text{span}\{1, \sin^2 x, \cos^2 x\} = \text{span}\{1, \sin^2 x\}$

e.g. $\{(1,2,1)\}$ is l.i. since $a(1,2,1) = (0,0,0)$
$\Rightarrow a = 0$

FACT If $v \in V$ is $\{v\}$ is l.i. $\Leftrightarrow$ $v \neq 0$.

e.g. $\{(0,0,0,0), (1,2,3,4)\}$ is l.d. since
$1(0,0,0,0) + 0.(1,2,3,4) = 0$

FACT Any set containing the zero vector is l.d.

FACT $\{u, v\}$ is l.d. $\Leftrightarrow$ $u = kv$ or $v = lu$
(if in $\mathbb{R}^3$ $\Leftrightarrow$ u, v are collinear)

FACT $\{u, v, w\}$ is l.d. $\Leftrightarrow$ $u \in \text{span}\{v, w\}$ or
$v \in \text{span}\{u, w\}$ or
$w \in \text{span}\{u, v\}$ $\star$

If u, v, w live in $\mathbb{R}^3$, then

$\{u, v, w\}$ l.d. $\Leftrightarrow$ u, v, w are coplanar (or collinear).

In general vector spaces can only say *

19. Lemma (4.10) A set $\{v_1, \dots, v_n\} \subset V$ is l.d. $\Leftrightarrow$ at least one of the vectors is a l.c. of the others. Pf: "$\Rightarrow$" If $a_1v_1 + \dots + a_nv_n = 0$, and $a_n \neq 0$ (say), then $v_n = -\frac{a_1}{a_n}v_1 - \frac{a_2}{a_n}v_2 - \dots - \frac{a_{n-1}}{a_n}v_{n-1}$

Note: $\{v_1, \dots, v_n\} \stackrel{l.d.}{\Longrightarrow}$ (say $v_n \in \operatorname{span}\{v_1, \dots, v_{n-1}\}$) then
$v_n = b_1v_1 + \dots + b_{n-1}v_{n-1}$
$\operatorname{span}\{v_1, \dots, v_n\} = \operatorname{span}\{v_1, \dots, v_{n-1}\}$.

Since $a_1v_1 + \dots + a_nv_n = (a_1 + a_nb_1)v_1 + \dots + (a_{n-1} + a_nb_{n-1})v_{n-1}$

N.B. This says a linearly dependent spanning set may be reduced in size i.e. Suppose $W = \operatorname{span}\{v_1, \dots, v_n\}$

If $\{v_1, \dots, v_n\}$ is l.d., (Suppose $v_n \in \operatorname{span}\{v_1, \dots, v_{n-1}\}$)

then $W = \operatorname{span}\{v_1, \dots, v_{n-1}\}$

e.g. $W = \operatorname{span}\{\overset{v_1}{(1,0,0)}, \overset{v_2}{(0,1,0)}, \overset{v_3}{(1,1,0)}\}$

But $v_3 = v_1 + v_2$, so $W = \operatorname{span}\{v_1, v_2\}$ (x-y plane)

e.g. Since $1 = \cos^2 x + \sin^2 x$ in $F[0, 2\pi)$,

$\operatorname{span}\{1, \sin^2 x, \cos^2 x\} = \operatorname{span}\{\cos^2 x, \sin^2 x\}$
$= \operatorname{span}\{1, \sin^2 x\}$.

We can decrease the size of a l.d. spanning set
How can we increase the size of a l.i. set? (20)

FACT If $\{v_1, \dots, v_n\}$ is l.i. in $v \in V$, then

$\{v, v_1, \dots, v_n\}$ is l.i. $\iff$ $v \notin \text{span}\{v_1, \dots, v_n\}$

["$\Rightarrow$" ✓; "$\Leftarrow$" Suppose $av + a_1v_1 + \dots + a_nv_n = 0$; if $a \neq 0$, $v \in$ sp. so $a = 0$. Then $a_1 = \dots = a_n = 0$; so $a = a_1 = \dots = a_n = 0$

i.e. we can increase the size of a l.i. set if we can find a vector not in its span

e.g. $\left\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$ is l.i.; $M_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin \text{span}\{M_1, M_2\}$,

($\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ has no soln.)

So $\{M_1, M_2, M_3\}$ is l.i.

e.g. $\{\overset{v_1}{(2,1,0)}, \overset{v_2}{(-1,0,1)}\}$ is l.i. (neither is a mult of other) ok for 2

Can we find $v_3 \in \mathbb{R}^3$ s.t. $v_3 \notin \text{span}\{v_1, v_2\}$? We know

$\text{span}\{v_1, v_2\} = \{v \in \mathbb{R}^3 \mid v_1 \times v_2 \cdot v = 0\}$ $\quad \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1, -2, 1)$

$= \{(x, y, z) \mid x - 2y + z = 0\} = S$

Choose $v = (1, 0, 0) \notin S$ e.s.

$\{(2,1,0), (-1,0,1), (1,0,0)\}$ is l.i.

(many choices)

eg $\{(2,1,0), (-1,0,1), (1,-2,1)\}$ is l.i.

e.g. $\{ \overset{v_1}{(1,0)}, \overset{v_2}{(0,1)} \}$ is l.i. in $\mathbb{R}^2$. Can we find $v_3 \in \mathbb{R}^2$ s.t. $\{v_1, v_2, v_3\}$ is l.i. ?

NO: since we know span$\{v_1, v_2\} = \mathbb{R}^2$!

$\therefore \forall v \in \mathbb{R}^2$, $\{v_1, v_2, v\}$ is <u>l.d.</u>

Same for $\{ \overset{w_1}{(1,-1)}, \overset{w_2}{(1,1)} \}$: $\{w_1, w_2, w\}$ is l.d. $\forall w \in \mathbb{R}^3$.

Indeed, if u_1, u_2 are any 2 non zero, non parallel v. in $\mathbb{R}^2$, $\{u_1, u_2, u\}$ is l.d. for any choice of $u \in \mathbb{R}^2$ since span$\{u_1, u_2$

<u>Claim</u> : Any 3 vectors in $\mathbb{R}^2$ are l.d. !

pf Suppose $\{v_1, v_2, v_3\}$ is l.i. in $\mathbb{R}^2$. Then, $\{v_1, v_2\}$ is l.i. (any subset of a l.i. set is l.i.) So, as before $\{v_1, v_2, v_3\}$ is l.d. for any $v \in \mathbb{R}^2$.

<u>FACT</u>: 4 or more vectors in $\mathbb{R}^3$ are l.d.

Thm 4.13 If a v.s. V can be spanned by n
<u>Lemma</u>

vectors, then any l.i. subset has <u>at most</u> n vectors. ($\Leftrightarrow$ if V has a subset of m l.i., any spanning set has <u>at least</u> m vectors)

Size of any spanning set for V $\geq$ size of any l.i. subset of V. (22)

(pf: later... $\mathcal{P}_2$ is spanned by $P_1 P_2 P_3$ so any 4 or more vectors are l.d.

e.g. • $\mathbb{R}^4$ is spanned by 4 vectors - so any 5 or more vectors in $\mathbb{R}^4$ are l.d.

• M_{22} is spanned by 4 vectors, so any 5 or more matrices in M_{22} are l.d.

• $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_{22} \mid a, b \in \mathbb{R} \right\}$ is spanned by 2 matrices, so any 3 or more matrices in D are l.d.

• $U = \{(x,y,z) \mid x+z=0\} = \operatorname{span}\{(1,0,-1), (0,1,0)\}$ is spanned by 2 vectors, so any 3 or more vectors in U are l.d. (while $\exists$ 3 l.i. in $\mathbb{R}^3$, ... eg $\{(1,0,-1), (0,1,0), (1,0,1)\}$ is l.i. in $\mathbb{R}^3$

not in U.

P. 304
P. 195

Basis & Dimension

Def^n A set $\{v_1, \dots, v_n\} \subset V$ is a basis of V if

(1) $\{v_1, \dots, v_n\}$ is l.i.

(2) " spans V

(largest l.i. set, smallest spanning set)

eg $\{(1,0), (0,1)\}$ is a basis of $\mathbb{R}^2$

$\{(1,1), (1,-1)\}$ "

Do all bases have same # of elts?

Thm 2 (P.304, P.195) If $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ are bases of V, then $n = m$.

Pf. Since $\{v_1, \dots, v_n\}$ spans V and $\{w_1, \dots, w_m\}$ is l.i.

$$n \geq m$$

" $\{w_1, \dots, w_m\}$ spans V and $\{v_1, \dots, v_n\}$ "

$$\underline{m \geq n.}$$

Def^n P.304 P.195 If V has a finite basis $\{v_1, \dots, v_n\}$, the dimension of V is n ("$\dim V = n$")

V is s.t.b finite dimensional.

e.g. $Sl_2 = \{ A \in M_{22} \mid \operatorname{tr} A = 0 \}$ $\operatorname{tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+d$ 24

$= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ (seen before

$= \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{M_3} \right\}$;

so $\{M_1, M_2, M_3\}$ spans Sl_2; moreover,

$aM_1 + bM_2 + cM_3 = 0 \Rightarrow \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a = b = c = 0$

$\therefore \{M_1, M_2, M_3\}$ is l.i.

$\therefore \{M_1, M_2, M_3\}$ is a basis of Sl_2

$\therefore \dim Sl_2 = 3$

---

e.g. $W = \operatorname{span}\{1, \sin x, \cos x\}$ (s.s. of $F[0, 2\pi]$)

$\{1, \sin x, \cos x\}$ spans W (by defn of W)

$\{1, \sin x, \cos x\}$ is l.i. (shown last class)

$\therefore \{1, \sin x, \cos x\}$ is a basis of W.

$\therefore \dim W = 3$

e.g. $V = \mathbb{R}^n$; $e_i = (0, 0, \ldots, 0, \underbrace{1}_{i\text{th position}}, 0, \ldots, 0)$ $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$

eg $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of D l.i. ✓ span ✓

$\therefore \dim D = 2$

eg $U = \{ (x, y, z) \mid x + z = 0 \}$

$= \operatorname{span} \{ \overset{v_1}{(1, 0, -1)}, \overset{v_2}{(0, 1, 0)} \}$; $\{ (1, 0, -1), (0, 1, 0) \}$ is also l.i. $\therefore \{ v_1, v_2 \}$ is a basis of U. $\therefore$

$\dim U = 2$

eg $\dim M_{22} = 4$ since $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, [\;\;] \ldots \right\}$ is a basis

$\dim M_{mn} = mn$ since $\{ E_{ij} \in M_{mn} \mid E_{ij}$ has 1 in i^{th} row & j^{th} col, zero elsewhere $\}$ is a basis

e.g. $F(\mathbb{R}) = \{ f : f : \mathbb{R} \to \mathbb{R} \}$ does not have finite dimension since, $\{ 1, x, x^2, \ldots, x^n \}$ is l.i. for any $n \in \mathbb{N}$: Suppose $p(x) = a_0 1 + a_1 x + \cdots + a_n x^n = 0$ for all $x \in \mathbb{R}$. Then $p(x)$, a polynomial of degree (at most n) has more than n roots! Impossible $\therefore a_0 = a_1 = \cdots = a_n = 0$.

(Understand...) [illegible]

Remark

RECALL

Size of any spanning set for V $\geq \dim V \geq$ size of an l.i. set $*$ (26)

How can we find bases for V?

Thms 3.4 Pps 306-307 (197-198) Suppose $\dim V = \underline{n}$ $(<\infty)$. Then

(1) Any l.i. set $\{v_1, \dots, v_{\underline{n}}\} \subset V$ is a basis (i.e. spans too!)

(2) Any spanning set $\{w_1, \dots, w_{\underline{n}}\} \subset V$ " (i.e. is l.i. too)

(3) If $\{v_1, \dots, v_m\}$ is l.i., there are $v_{m+1}, \dots, v_n$ s.t. $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ is a basis (every l.i. set can be extended to a basis)

(4) If $S = \{w_1, \dots, w_\ell\}$ spans V, there is a subset of S that is a basis of V.

Pf 1) If $\{v_1, \dots, v_n\}$ didn't span V, $\exists v \in V$ s.t. $\{v, v_1, \dots, v_n\}$ is l.i., contradicting $*$

2) If $\{w_1, \dots, w_n\}$ is l.d., then (say) $w_1 \in \operatorname{span}\{w_2, \dots, w_n\}$; so

$V = \operatorname{span}\{w_1, \dots, w_n\} = \operatorname{span}\{w_2, \dots, w_n\}$ is spanned by $(n-1)$ vectors, contradicting $*$.

3) ...
4) ... } later;

e.g. (for 1) $U = \{(x, y, z) \mid x + z = 0\}$; know $\dim U = 2$.

Since $(1, 1, -1)$ & $(0, 1, 0)$ both belong to U and are l.i.,

$\{(1, 1, -1), (0, 1, 0)\}$ is another basis for U.

e.g. for (3.) Note: $U \subset \mathbb{R}^3$, $\{\overset{v_1}{(1, 1, -1)}, \overset{v_2}{(0, 1, 0)}\}$ is l.i. Let's extend this (basis of $\underline{U}$) to a basis of $\underline{\mathbb{R}}^3$: need $v \notin U$ e.g. $v = (1, 0, 0)$.

Then, we know $\{v, v_1, v_2\}$ is l.i. Since $\dim \mathbb{R}^3 = 3$, $\{v, v_1, v_2\}$ is a basis of $\mathbb{R}^3$.

What if $\dim W = n$ and $U \subseteq W$ is a subspace?

Thm 5 (P. 307) If $\dim W = n$, and U is a s.s. of W then 1) $\dim U \le \dim W$

2) $\dim U = \dim W \iff U = W$.

Pf. 1) take a basis of U, $\{u_1, \dots, u_m\}$, extend* to a basis of W. Then since its l.i., $m \le n = \dim W$ (since spanning set). So $\dim U \le \dim W$

2) If $\{u_1, \dots, u_n\}$ is a basis of U; so its l.i. Since $\dim W = n$ also, by 4.14-4.16 (1), its a basis of W. But $U = \operatorname{span}\{u_1, \dots, u_n\} = W$, so $U = W$.

e.g. $U = \{(x,y,z) \mid x+z=0\}$ $\dim U = 2$, $\{(1,1,-1), (0,1,0)\}$ basis; can extend to a basis of $\mathbb{R}^3$: need $v \notin U$, $v = (1,0,0)$...

e.g. Any 3-dim'l subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$

Corollary If $\{u, v, w\}$ is non-coplanar, then $\{u, v, w\}$ is a basis of $\mathbb{R}^3$. ($\{u,v,w\}$ non-coplanar $\iff$ $\{u,v,w\}$ l.i. $\iff$ $u \times v \cdot w \ne 0$ see soon $\Leftarrow$ v. easy.)

e.g. Any 4-dim'l subspace of M_{22} is M_{22}

e.g. Any 2-dim'l ss of $U = \{(x,y,z) \mid x+z = 0\}$ is U.

<u>What are bases good for?</u> (5.4.3)

Suppose $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis of V; Then for every $v \in V$ there are <u>unique</u> scalars $x_1, \dots, x_n \in \mathbb{R}$ s.t.

$$v = x_1 v_1 + x_2 v_2 + \dots + x_n v_n.$$

– exist: $\mathcal{B}$ spans V

– unique because $\mathcal{B}$ l.i.

$$(v = y_1 v_1 + \dots + y_n v_n$$
$$\Rightarrow 0 = (x_1 - y_1) v_1 + \dots + (x_n - y_n) v_n)$$
$$\Rightarrow x_i = y_i,$$

Fix $\mathcal{B} = \{v_1, \dots, v_n\}$ (28) (29)

idea $\qquad V \qquad\qquad\qquad\qquad \mathbb{R}^n$

$$v = x_1 v_1 + \dots + x_n v_n \longleftrightarrow (x_1, \dots, x_n)$$

$$M_{22} \qquad\qquad\qquad \mathbb{R}^4$$

$\mathcal{B}$ = Standard basis $\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow (a, b, c, d)$

$$SL_2 \qquad\qquad\qquad \mathbb{R}^3$$

$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \qquad \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \longleftrightarrow (a, b, c)$

All vector spaces of dimension n
(no matter how bizarre looking)
behave exactly like $\mathbb{R}^n$!

1.1 – done in examples

1.2 Linear Systems

(8×10^6 eqns, 5×10^6 variables; air flow $\vec{v}, T, \rho$; 100^3 nodes) (30)

unknowns

e.g.

$$* \begin{cases} x_1 + x_2 + x_4 = 1 \\ x_3 - x_4 = -1 \end{cases}$$

2 eqns, 4 unknowns

coefficients; RHS (constant term)

$(1,0,0,0)$ isn't a soln of $*$ (doesn't satisfy 2nd eqn)

$(2,0,-2,-1)$ is a soln of $*$

- General soln to a system is the set of all solns:

Claim: $S = \{(1-s-t, s, t-1, t) \mid s,t \in \mathbb{R}\}$ is the general soln to $*$: check (I) $(1-s-t, s, t-1, t)$ is a soln of $*$ for every $s,t \in \mathbb{R}$

$x_1 + x_2 + x_4 = (1-s-t) + s + t = 1$ ✓

$x_3 - x_4 = (t-1) - t = -1$ ✓

(II) Every soln of $*$ is in S: Use first eqn to write $x_1 = 1 - x_2 - x_4$

2nd $x_3 = -1 + x_4$

$\therefore$ (x_1, x_2, x_3, x_4) is a soln to $*$ $\Leftrightarrow$ $(x_1, x_2, x_3, x_4) = (1-x_2-x_4, x_2, -1+x_4, x_4)$

* Change names of x_2, x_4 to s, t to obtain S above.

s, t are called the parameters in gen'l soln.

Note: $*$ has ∞ly many solns

e.g.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + x_3 &= 5 \\ x_3 &= 1 \end{aligned}$$

3×3 system. (3 planes)

has a unique soln (only one) $(-7, 4, 1)$

1.4 Matrix Multiplication

- useful for
  - probability (transition matrices)
  - economics (Leontief)
  - geometry
  - quantum theory
  - solving linear systems
  - vector spaces (keeping track of l.c.)
  - computing your final grade.

matrices
- arrays of #'s
- collection of rows
- collection of cols
- "gen

Recall $m \times n$ matrix A : A has m rows (n entries), n cols. (m entries)

notation $A = [a_{ij}]$, $a_{ij} = ij^{th}$ entry $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [a_{ij}]$: $a_{11} = 1$, etc.

Def^n (P.27) If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then AB is the $m \times p$ matrix whose (i, j) entry is the dot product of the i^{th} row of A and the j^{th} col. of B.

i.e. if $A = [a_{ij}]$ & $B = [b_{ij}]$ then

$$AB = [c_{ij}] \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = [a_{i1} \cdots a_{in}] \cdot \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Remark AB makes sense iff # cols A = # rows of B.

e.g.
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} \stackrel{!}{=} x \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + z \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

l.c. of cols of ( ).

$$[1 \ 2 \ 3] \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = [a + 2d + 3g \quad b + 2e + 3h \quad c + 2f + 3j] \stackrel{!}{=} 1 \cdot [a \ b \ c] + 2 [d \ e \ f] + 3 [g \ h \ j]$$

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$$= \begin{bmatrix}1 & 2\\ 2 & 6\\ 1 & -1\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\end{bmatrix} = \begin{bmatrix}4 & -1 & 10 & 5\\ 8 & -4 & 26 & 12\\ 4 & 2 & 1 & 2\end{bmatrix}$$

$$\underset{3\times 2}{A} \qquad \underset{2\times 4}{B} \qquad \underset{3\times 4.}{AB}$$

BA doesn't make sense here, "isn't defined"!

e.g. $\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\ 5\\ -6\end{bmatrix} = \begin{bmatrix}-4\end{bmatrix} = -4 = (1,2,3).(4,5,-6)$

convention (remove br.)

dot product.

i.e. if vectors are written as rows, $u \cdot v = uv^t$

dot product   matrix product.

cols $u \cdot v = u^t v$

dot

e.g. $\underset{w}{\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}} \cdot \underset{x}{\begin{pmatrix}4\\ 5\\ -6\end{pmatrix}} = \underset{w^t}{\begin{bmatrix}1 & 2 & 3\end{bmatrix}}\underset{x}{\begin{bmatrix}4\\ 5\\ -6\end{bmatrix}} = -4$

matrix produ

$w \cdot x = w^t x$ (cols)

e.g. $\underset{\substack{3\times 1\\ A}}{\begin{bmatrix}4\\ 5\\ -6\end{bmatrix}}\underset{\substack{1\times 3\\ B}}{\begin{bmatrix}1 & 2 & 3\end{bmatrix}} = \underset{\substack{3\times 3!\\ AB}}{\begin{bmatrix}4 & 8 & 12\\ 5 & 10 & 15\\ -6 & -12 & -18\end{bmatrix}} \neq \underset{BA}{\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\ \xi\end{bmatrix}} = -4$

also defined

$\downarrow$ L14

TWO INTERESTING FACTS ABOUT Matrix mult

① Matrix mult^n is <u>not</u> commutative

(P28) i.e. AB is not always the same as BA (even if they have the same size)

e.g.

$$\overset{A}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\neq$$

$$\overset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$AB \neq BA$ (<u>always</u>)

(turns out to be <u>very</u> useful)

②
$$\underset{C}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$CD = 0 \not\Rightarrow C = 0 \text{ or } D = 0 \,!$$

Recall if $A = [a_{ij}]$ is $m \times n$

then $A^t = [b_{ij}]$ where $b_{ij} = a_{ji}$

1.1.4 <u>Transpose</u>

Already discussed

$(A+B)^t = A^t + B^t$ | $(A^t)^t = A$

Summary of properties (Thm 1, 3 pps 29–)

Let A, B, C be matrices, k a scalar, I. Whenever defined,

1. $(AB)C = A(BC)$
2. $A(B+C) = AB + AC$
3. $(B+C)A = BA + CA$
4. $k(AB) = (kA)B = A(kB)$
5. $(AB)^t = B^t A^t$ ← **N.B.** reversal of order
6. $AI = A$, $IB = B$

5×3

$A \quad B$
3×6 × 6×5

A^t
6×6 6×3

Remark See the web to discover how matrix multn is used in your final grade calculation

$$x = \begin{bmatrix} d \\ mt1 \\ mdt \\ mt2 \\ fe \end{bmatrix} \qquad T = \begin{bmatrix} \frac{1}{12} & & & & \\ & \frac{1}{22} & & & \\ & & \frac{1}{24} & & \\ & & & \frac{1}{22} & \\ & & & & \frac{1}{48} \end{bmatrix}$$

W = 5×16 "possible weights" matrix

$$= \begin{array}{c} \begin{bmatrix} 0 & & .1 & & .1 \\ 0 & & 0 & & .1 \\ 0 & \cdots & .2 & \cdots & .2 \\ 0 & & .1 & & .1 \\ 1 & & .6 & & .5 \end{bmatrix} \end{array} \begin{array}{l} d \\ mt1 \\ mdt \\ mt2 \\ fe \end{array}$$

$$B = \underset{16\times5\ \ 5\times5\ \ 5\times1}{W^t T x} = 16\times1 \text{ matrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_{16} \end{bmatrix}$$

$f = \max\{B_1, \ldots B_{16}\}$.

"Square matrices : can multiply as numbers
of fixed size

e.g. P35 $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ $A^2 = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$, $A^3 = \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$

interesting fact (2×2'' for now)

$\operatorname{tr} A = -3$

$\det A = -10$

$$A^2 - \operatorname{tr}(A)A + \det A \cdot I_2$$

$$= \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - 10 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} !$$

~~Inverses: defer~~

exercise: show true for any 2×2.

1.4.4 Block mult'n

(can sometimes (carefully) treat parts of matrices as if they were numbers!)

e.g. $A = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$

1 entry: 5 mult

A^2: 25 × 5 = 125 mult.

1 terahertz
$= 10^{12}$

$\frac{10^{100} \times 125}{10^{12}}$ secs

$= 10^{100} \times 1.25$

$\approx \frac{10^{99} \times 1.25}{\pi \times 10^8}$ yrs

$\approx 4 \times 10^{81}$ yrs

Then $A^2 = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} B^2 & 0 \\ 0 & C^2 \end{bmatrix}$ works!

$A^{10^{100}} = \begin{bmatrix} B^{10^{100}} & 0 \\ 0 & C^{10^{100}} \end{bmatrix}$; $B^{10^{100}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

1GHz = 10^9

10^{100} 125 10^{90} $C^{10^{100}} = 0$! $A^{10^{100}} = \begin{bmatrix} I_3 & 0 \end{bmatrix}$

1.4.3 Matrix eqn of a linear system, etc (more block multn appln)

e.g.

Sys
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ x_2 + x_3 &= -1 \\ x_1 + \quad + x_3 &= 4 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{b}$$

matrix eqn of Sys

$$\boxed{{}_m[\overset{n}{A} \mid b] \quad : \quad Ax = b}$$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ A $m \times n$, b $m \times 1$, x $n \times 1$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Saw before: use block multn w/o knowing

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [c_1 \; c_2 \; c_3]; \qquad c_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad c_2 = \ldots \quad c_i = i\text{th col of } A$$

$x = \begin{bmatrix} x \\ 2 \\ 3 \end{bmatrix}$

$$Ax = [c_1 \; c_2 \; c_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 c_1 + x_2 c_2 + x_3 c_3$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\in \operatorname{span}\{c_1, c_2, c_3\}$$

In general

A $m \times n$; $\quad A = [c_1 \cdots c_n) \qquad c_i = i\text{th col. of } A,\; c_i \in \mathbb{R}^m$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$Ax = [c_1 \cdots c_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 c_1 + \cdots + x_n c_n$$

Consequences: $(A|b)$

FACT (1) $Ax = b$ is consistent $\iff$ b is a l.c. of cols of A

(2) $\underset{(m\times n)}{A}x = 0$ has a unique sol^n $(x = 0)$ $\iff$ the columns of A are l.i. $\iff$ rank $A = n$.

$\left(Ax = [c_1 \cdots c_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1c_1 + \cdots + x_nc_n = 0 \right.$ has a unique sol^n $x_1 = \cdots = x_n = 0 \iff \{c_1, \dots, c_n\}$ is l.i. $)$

e.g. $Ax = b$: $\begin{array}{cccc} c_1 & c_2 & c_3 & b \end{array}$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

is consistent since rank$(A|b)$ = rank $A = 2$. $\therefore (2,-1,4) \in \text{span}\{(1,0,1),(2,1,0),(3,1,1)\}$

(e.g. $b = 3c_1 - 2c_2 + c_3$, $b = 4c_1 - c_2$)

e.g. ("homog." sys) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\therefore$ does not have a unique sol^n since rank $A = 2 < 3 = \#$ cols of A. $\therefore$ cols of A are l.d.

$\therefore \{(1,0,1),(2,1,0),(3,1,1)\}$ is l.d.

i.e. $c_3 = c_1 + c_2$

$\mathbb{R}^n$ ... are l.i. $\iff$ $[v_1\ v_2 \cdots v_n] = n$!

1.4 Homogeneous Systems

Suppose A is $m \times n$

$Ax = 0$ clearly has $x = 0$ as soln

Defn $\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$ is the kernel of A (null space)

P.181 (4.1) Thm For any $m \times n$ matrix A, $\ker A$ is a subspace of $\mathbb{R}^n$.

Pf. Let $Ax = 0$ be the matrix eqn of the system. The $\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$ satisfies ① $0 \in \ker A$ since $A0 = 0$

② If $u, v \in \ker A$ and $k \in \mathbb{R}$

$$A(u + kv) = Au + kAv = 0 + k.0 = 0$$

Hence $u + kv \in \ker A$ Thus $\ker A$ is subspace of $\mathbb{R}^n$.

e.g. $Av = 0$ is
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim .. \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Gen'l soln is $x = -s$, $y = -s$, $z = s$; $s \in \mathbb{R}$; ie $\ker A = \{v \in \mathbb{R}^3 \mid Av = 0\}$

is $\{(-s, s, s) \mid s \in \mathbb{R}\}$

$= \text{span}\{(-1, 1, 1)\}$! *

(line through origin with dirn $(-1, 1, 1)$)

basis of $\ker A = \{(-1, 1, 1)\}$ (spans by * l.i. since $(-1,1,1) \neq 0$)

$\dim \ker A = 1$

e.g. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right] = [A \mid b]$ rank $A = 2 < 3 = \#$variables

$\therefore$ will have ∞ly many solns

Theorem 1.7 (P.22, 24) A homogeneous system $Ax = 0$ with fewer eqns than unknowns always has ∞ly many solns; the number of parameters in the gen'l soln = #variables − rank A.

Pf. A is $m \times n$ so $m < n$ is the assumption; then

rank $A \le m$ (& rank $A \le n$) so   rank $A \le m < n$ $\therefore n - \text{rank} A > 0$

$\therefore$ #params $= n - \text{rank} A > 0$. $\therefore$ at least 1 param in genl soln.

Remark This is the heart of the fundamental fact

size of a spanning set $\ge$ size of a l.i. set
(4.4.3, P.200)

Recall $\underset{\text{has unique soln}}{Ax = 0} \iff$ cols of A are l.i. $A = [v_1 \cdots v_n]$ $v_i \in \mathbb{R}^m$

If $m < n$ $Ax = 0$ never has a unique soln so cols of A are always l.d.

i.e. if $m < n$, n vectors in $\mathbb{R}^m$ are always l.d.

So $\ker A = \{x \mid Ax = 0\}$ is a subspace of $\mathbb{R}^n$; what is its dimension?

e.g. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 2 & 2 & 3 & 7 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} & s & & t \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}$
$x_1 = -s - 2t$
$x_2 = s$
$x_3 = -t$
$x_4 = t$
$s, t \in \mathbb{R}$

$\ker A = \{(-s-2t, s, -t, t) \mid s, t \in \mathbb{R}\} = \text{Span}\{\overset{v_1}{(-1,1,0,0)}, \overset{v_2}{(-2,0,-1,1)}\}$

So $\{v_1, v_2\}$ spans $\ker A$. Moreover, $sv_1 + tv_2 = 0 \iff s = t = 0$

$\therefore \{v_1, v_2\}$ is a basis of $\ker A$ $\therefore \dim \ker A = 2$

$= $ # parameters!
N.B.

① $A = [c_1 \cdots c_n]$ $\quad x \in \mathbb{R}^n \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ (48e)

$(Ax = 0 \Rightarrow x = 0) \Leftrightarrow c_1, \ldots, c_n$ are l.i.

$\Updownarrow \qquad\qquad \Rightarrow$

rank $A = n$

② $Ax = b$ is consistent $\Leftrightarrow$ $b \in \operatorname{span}\{c_1, \ldots, c_n\}$

③ $\ker A = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \subset \mathbb{R}^n$

ss of $\mathbb{R}^n$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 2 & 2 & 3 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -s - 2t \\ x_2 = s \\ x_3 = -t \\ x_4 = t \end{array}$$

$= \{ (-s-2t, s, -t, t) \mid s,t \in \mathbb{R} \}$

$\ker A = \operatorname{span}\{ \underset{v_1}{(-1,1,0,0)}, \underset{v_2}{(-2,0,-1,1)} \}$ "basic" solutions

$sv_1 + tv_2 = 0 \Leftrightarrow s = t = 0$

$\{v_1, v_2\}$ basis for $\ker A$; $\dim \ker A = \#$ param $= 2$

Sol^n obtained this way are sometimes called the basic sol^{ns}

4.6.1 Theorem 2+6 (P.209) The set of basic sol^{ns} (P.25) of a homogeneous linear system $AX=0$ is a basis of the set of all solutions (ker A), which has dimension n − rank A. (If A is $m \times n$.)

Rmk ← See opposite (pf of lemma 4.4.3 P200)

25/10/02 (16/11/09, 19/3/02)

1.5 Matrix Inverses (for square matrices only)

P.38

There's a profit in comparing $Ax = b$ (A $m \times n$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$) to $ax = c$; $a, x, c \in \mathbb{R}$,

If $a \neq 0$, $x = c/a$ (is the unique sol^n). Can we "divide by A?" Careful: $AB = C$

$B = \cancel{\frac{C}{A}} = A^{-1}C$ or CA^{-1}

Recall $I_n = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = $n \times n$ diagonal matrix with n 1's.

Def^n (P.39) If A is an $n \times n$ matrix and B is an $n \times n$ matrix s.t. $AB = BA = I$, then B is called an inverse of A, and we write $B = A^{-1}$. A is then called invertible.

Rmk: ... ~~(P47, Cor 1)~~ If $AB = I$, then $BA = I$ too. See web eg. $I_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ is invertible, $I_n^{-1} = I_n$.

e.g. $\underset{A}{\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}} \underset{B}{\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so A, B are both invertible

$A = B^{-1}$, $B = A^{-1}$

eg $\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}$ is not invertible since $\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \underset{x_0}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

$(x_0 \neq 0)$

(if A^{-1} existed $x_0 = A^{-1}(Ax_0) = A^{-1}(0) = 0$).

FACT

(P.40) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Leftrightarrow$ $ad - bc \neq 0$;

If $ad - bc \neq 0$, $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ check.

(checking formula works is easy; ex. show $ad - bc = 0 \Rightarrow A$ is not invertible)

$A \begin{bmatrix} d \\ -c \end{bmatrix}$.

(1.5.4)

Some Facts ① (Thm 3, P43) If $a \neq 0$ is a scalar, $k \in \mathbb{N}$, A & B are invertible

then so is

1) A^{-1} and $(A^{-1})^{-1} = A$

2) A^k and $(A^k)^{-1} = (A^{-1})^k$

3) A^t and $(A^t)^{-1} = (A^{-1})^t$

4) (aA) and $(aA)^{-1} = a^{-1}A^{-1}$

5) AB and $(AB)^{-1} = B^{-1}A^{-1}$

1.5.2 Applications to linear systems

Thm 2 1.4 If A is an invertible $n \times n$ matrix, then for any $b \in \mathbb{R}^n$, the linear system $Ax = b$

1) is always consistent
2) has the unique sol^n $x = A^{-1} b$

pf. 1) Check: $A \cdot (A^{-1}b) = (AA^{-1})b = I \cdot b = b$ so $x = A^{-1}b$ is a sol^n

2) Suppose $Ax' = b$, then $A^{-1}(Ax') = A^{-1}b \Rightarrow (A^{-1}A)x' = A^{-1}b \Rightarrow x' = b^*$, so $A^{-1}b$ is the only sol^n.

Corollaries If A is invertible, $A = [c_1 \cdots c_n]$

1) The columns of A span $\mathbb{R}^n$ (since $Ax = b$ is always consistent)
2) The columns of A are lo lo (since $Ax = 0$ has the uniq soln $0 = A^{-1}0$)

1) + 2) $\Rightarrow$ 3) The cols of A are a basis of $\mathbb{R}^n$

L17 1.5.3

Finding Inverses (when they exist)

Note A^{-1} exists $\Rightarrow$ rank $A = n$

$\Rightarrow$ RRE form of A is I_n

$\overset{?}{\Leftarrow}$

YES!

e.g. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 2 & 5 \end{bmatrix}$;

we solve $AC = I_3$ for C. Write $C = [c_1\ c_2\ c_3]$

$I_3 = [e_1\ e_2\ e_3]$ $\qquad e_i, c_i$ ith col of I_3, C resp.

Then $AC = A[c_1\ c_2\ c_3] = [Ac_1\ \ Ac_2\ \ Ac_3] = I_3 = [e_1\ e_2\ e_3]$

is equivalent to
$$Ac_1 = e_1$$
$$Ac_2 = e_2$$
$$Ac_3 = e_3$$

Solve $[A \mid e_1] \overset{①}{\sim} [\tilde{A} \mid \tilde{e}_1] \overset{?}{\sim} [I_3 \mid c_1]$ Note: the row operations used in each are completely determined by A,

$[A \mid e_2] \overset{②}{\sim} [\tilde{A} \mid \tilde{e}_2] \sim [I_3 \mid c_2]$

$[A \mid e_3] \overset{③}{\sim} [\tilde{A} \mid \tilde{e}_3] \sim [I_3 \mid c_3]$

so are the same in each of ①, ②, ③ : Do them all at once!

Use "superaugmented" matrix $[A \mid e_1\ e_2\ e_3] \doteq [A \mid I_3] \overset{?}{\doteq} [I_3 \mid$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 1 & 0 \\ 2 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 5 & 0 & -2 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \sim \begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \\ \begin{array}{cccc} \text{unique soln!} & c_1 & c_2 & c_3 \end{array} \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{bmatrix}$$

Thm: If A is an $n \times n$ matrix and rank $A = n$, then A is invertible, and A^{-1} can be computed via the algorithm above i.e. $[A \mid I_n] \sim [I_n \mid A^{-1}]$

A is not invertible

Remark ①: The algorithm decides if A^{-1} exists, since it tells you rank A:

eg $[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 1 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -3 & -6 & -1 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & \\ 0 & 1 & 2 & * \\ 0 & 0 & 0 & \end{array}\right]$; rank $A < 3$ so A is not invertible.

② If $AC = I$ (A, C $n \times n$ matrices) then $CA = I$ too!

Pf: Consider $Cx = 0$. Then $ACx = A \cdot 0 = 0$. But $ACx = Ix = x$, so $Cx = 0 \Rightarrow x = 0$. Hence rank $C = n$, so by our algorithm, we can find B s.t.

$$CB = I.$$

Then

$$\underbrace{AC}_{I} B = A \cdot I$$

$$I \cdot B = A$$

$$B = A$$

i.e. $C \cdot A = I$

A, C $n \times n$ matrices

Remark : If $AC = I$, then $CA = I$ too: 54

Pf: If $AC = I$ ~~so if~~ then if $Cx = 0$, $ACx = Ix = x = 0$ so
$Cx = 0 \Rightarrow x = 0$; so rank $C = n$. Hence
can solve $CB = I$* for B! Then

$$\underbrace{AC}B = AI$$
$$I \cdot B = A$$
ie. $B = A$
so * says $CA = I$.

---

e.g. $\{(1,1,2),(1,2,5),(2,5,5)\}$ basis of $\mathbb{R}^3$
since matrix with these rows is in

$\{(1,2,3),(4,5,6),(1,-1,-3)\}$ isn't

$\{(1,4,1),(2,5,-1),(3,6,-3)\}$ isn't . . .

$\{(1,1,2)(1,2,2),(2,5,5)\}$ is . — —

Summary (Thm 5, P46 + Thm 2 P192 : 4.2.2)

The following are equivalent for an $n \times n$ matrix A

1) A is invertible $\Downarrow$ ✓

2) $AX = 0 \Rightarrow X = 0$ $\Downarrow$ ✓

3) rank $A = n$ $\Downarrow \cdots$

4) $A \sim I_n$ $\Downarrow$ ✓ (algorithm)

5) $AX = b$ is consistent for all $b \in \mathbb{R}^n$

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

$1 \Leftrightarrow 6$

$2 \Leftrightarrow 2^a$

$6, 2^a \Leftrightarrow 2^b$

$6, 5^a \Leftrightarrow 5^b$

$2^a), 5^a \Leftrightarrow 7$

$2^b), 5^b) \Leftrightarrow 8$

6) A^{-1} is invertible $3)^b$ rank $A^t = n$

2^a) the columns of A are l.i. 2^b) the rows of A are l.i.

5^a) the columns of A span $\mathbb{R}^n$ 5^b) the rows of A span $\mathbb{R}^n$

7) the columns of A are a basis for $\mathbb{R}^n$

8) the rows of A are a basis for $\mathbb{R}^n$

Rmk Rank A = Rank A^t!

e.g. $\{(1,1,2), (1,2,5), (2,5,5)\}$ is a basis of $\mathbb{R}^3$

since $\operatorname{rank}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 5 \end{bmatrix} = 3$

$\{(1,2,3), (4,5,6), (1,-1,3)\}$ isn't since

$\operatorname{rank}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -1 & -3 \end{bmatrix} < 3$

$\{(1,4,1), (2,5,-1), (3,6,-3)\}$ isn't . . .

$\{(1,1,2), (1,2,2), (2,5,5)\}$ is ✓

Finding bases for subspaces W of $\mathbb{R}^p$

3 problems (I) Given a spanning set $\{w_1, \ldots, w_m\}$ of W, find any basis of W

(II) Given a spanning set $\{w_1, \ldots, w_m\}$ of W, find a basis of W which is a subset of $\{w_1, \ldots w_m\}$

(III) Given a basis B of $W \subset \mathbb{R}^p$, extend B to a basis of $\mathbb{R}^p$.

We'll use matrices, and elem. row ops extensively.

Given an $m \times n$ matrix A, there are 2 (new) subspaces associated to it

Write $A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$ $r_i \in \mathbb{R}^n$, i^{th} row of A

$A = [c_1 \ldots c_n]$ $c_j \in \mathbb{R}^m$ c_j j^{th} col of A

Defn $\text{Row}(A) = \text{row space of } A = \text{Span}\{r_1, \ldots, r_m\} \subset \mathbb{R}^n$

$\text{Col}(A) = \text{col space of } A = \text{span}\{c_1, \ldots, c_n\}. \subset \mathbb{R}^m$
$= \{Ax \mid x \in \mathbb{R}^n\}$

Consider problem (I): Given $\{w_1, \ldots, w_m\}$, a spanning set of $W \subset \mathbb{R}^p$, find any bases of W

Row space algorithm: Set $A = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$, (5.8)

(Lemma, p209)

So $\text{row}(A) = W$; consider effect of an elem row op^n on A:

$A \underset{\text{row op}}{\xrightarrow{\text{elem}}} A'$; rows of A' are l.c. of rows of A

$\begin{bmatrix} 1&3&3&4 \\ 1&-1&-1&0 \\ 2&1&1&3 \end{bmatrix} \overset{-R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1&3&3&4 \\ 0&-4&-4&-4 \\ 2&1&1&3 \end{bmatrix}$ $\therefore \text{row}(A') \subseteq \text{row}(A)$

BUT! row op^n are reversible,

$A' \underset{\text{row op}}{\xrightarrow{\text{elem}}} A$ $\therefore$ rows of A are l.c. of rows of A'

$\begin{bmatrix} 1&3&3&4 \\ 0&-4&-4&-4 \\ 2&1&1&3 \end{bmatrix} \overset{R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1&3&3&4 \\ 1&-1&-1&0 \\ 2&1&1&3 \end{bmatrix}$ $\therefore \text{row}(A) \subseteq \text{row}(A')$

$\therefore \text{row}(A') = \text{row}(A) = W$

So take A to RE or RRE form: get "simplest" spanning set for W (will be l.i.) ; $W = \text{span}\{(1,3,3,4), (1,-1,-1,0), (2,1,1,3)\}$;

(RE)

e.g. set: $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1&3&3&4 \\ 1&-1&-1&0 \\ 2&1&1&3 \end{bmatrix} \sim \begin{bmatrix} 1&3&3&4 \\ 0&-4&-4&-4 \\ 0&-5&-5&-5 \end{bmatrix} \sim \begin{bmatrix} 1&3&3&4 \\ 0&1&1&1 \\ 0&0&0&0 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&1 \\ 0&1&1&1 \\ 0&0&0&0 \end{bmatrix}$

$\text{row}(A) = W$; $= A'$, $\text{row}(A') = W$; $= A''$, $\text{row}(A'') = W$; $= \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{bmatrix} = \tilde{A}$

$W = \text{row}(A) = \text{row}(\tilde{A}) = \text{span}\{\tilde{r}_1, \tilde{r}_2\} = \text{span}\{\underset{\tilde{r}_1}{(1,0,0,1)}, \underset{\tilde{r}_2}{(0,1,1,1)}\}$

$a\hat{r}_1 + b\hat{r}_2 = 0 \implies$

$$\begin{array}{cccc} (a & 0 & 0 & a) \\ (0 & b & b & b) \; + \\ \hline (a & b & * & *) \end{array}$$

$= \quad \implies a = b = 0.$

Lemma 1(1) P203
Thm1 P204. - , Row operations do not change the row space of a matrix. Moreover

1) the non-zero rows of a matrix in (RE) RRE form a basis for row(A)

2) rank A = dim row (A)

e.g. Find a basis for $W = \text{span}\{(1,3,7), (1,-1,5), (2,1,1)\}$

$$\text{Set } A = \begin{bmatrix} 1 & 3 & -7 \\ 1 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & 12 \\ 0 & -5 & 15 \end{bmatrix} \sim \overset{RE}{\begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}} \sim \begin{bmatrix} 1 & 0 & 16 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

cd stop here.

$CW = \text{row}(A')$

$\{(1,3,7), (0,1,-3)\}$ OR $\{(1,0,16), (0,1,-3)\}$

is a basis for W.

Remark: $\text{col}(A) \neq \text{col}(\tilde{A})$: $(1,1,2) \in \text{col}(A)$, but $(1,1,2) \notin \text{col}(\tilde{A}) =$ x-y plane

Problem II Given a spanning set $\{w_1, \ldots, w_m\}$ of a subspace of $\mathbb{R}^n$, find a basis which is a subset of the given spanning set.

$$\text{eg } W = \text{span}\{\overset{w_1}{(1,1,1,0)}, \overset{w_2}{(1,1,2,1)}, \overset{w_3}{(1,1,-1,1)}, \overset{w_4}{(1,1,2,2)}\} \qquad u_i = w_i^t$$

"Casting-out algorithm". Set $B = [u_1^* \; u_2^* \; u_3^* \; u_4^*]$

$u_j = j^{th}$ col of A

$\text{col}(B) = W$! First; reduce B to RE form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \ldots \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ \ldots \end{bmatrix} = B'$$

idea: recall $[A|b]$ consistent $\Leftrightarrow$ rank A = rank $[A|b]$ $\Leftrightarrow$ b is a l.c. of cols of A

We consider rank $[u_1]$, rank $[u_1\, u_2]$, rank $[u_1\, u_2\, u_3]$, etc.

- when rank increases, new col is not dependent on previous (add to basis)
- when rank doesn't increase – new col is dependent on previous cols (don't add to basis)

(rank increases whenever a new leading one occurs!)

$[u_1] = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; rank $[u_1] = 1$; add

$[u_1\, u_2] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (first 2 cols of B') rank $[u_1 u_2] = 2 >$ rank $[u_1]$

$\therefore \{u_1, u_2\}$ l.i.

$[u_1\, u_2\, u_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (1^{st} 3 cols of B') rank $[u_1 u_2 u_3] >$ rank $[u_1 u_2]$

$\therefore \{u_1, u_2, u_3\}$ l.i.

$B = [u_1\, u_2\, u_3\, u_4] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (B') rank $[u_1 u_2 u_3 u_4]$ = rank $[u_1 u_2 u_3]$

$\therefore u_4 \in$ span $\{u_1, u_2, u_3\}$

$\therefore$ basis of $col(B) = W$ is $\{u_1, u_2, u_3\}$

Note: these are the cols of B where the leading ones occur

~~Last class~~: (I) find *any* basis of $W = \text{span}\{w_1, \dots, w_m\} \subseteq \mathbb{R}^n$

- Rowspace algorithm: set $A = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$ w_i = i^{th} row of A
  - $W = \text{row}(A)$
- $A \sim A'$ (REch form of A); basis of W = basis of row(A) = row(A')
  = non-zero rows of A'
- $\dim W = \text{rank } A = \dim \text{row}(A)$

(II) find a basis of $W = \text{span}\{w_1, \dots, w_m\} \subseteq \mathbb{R}^n$ that is a subset of $\{w_1 \dots w_m\}$.

Casting out algorithm   set $B = [w_1 \cdots w_m]$ (w_j = j^{th} col of B)
$\text{col}(B) = W$

$\sim$ B' = REch form of B; If leading ones occur in columns $j_1, \dots, j_r$ of B', then cols $j_1, \dots, j_r$ of B are a basis of W.   • $\dim W = \dim \text{col}(B) = \text{rank } B$!

pblm (4.43) eg $W = \text{span}\{u_1, u_2, u_3, u_4, u_5\}$ $B = [u_1\ u_2\ u_3\ u_4\ u_5]$

$$B = \begin{array}{c} \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 3 & 7 & 7 \\ 1 & 2 & 2 & 5 & 5 \\ 3 & 6 & 6 & 15 & 14 \end{bmatrix} \\ u_1\ u_2\ u_3\ u_4\ u_5 \end{array} \sim \dots \sim \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B'$$

$$\sim \dots \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

		Building Basis
$\text{rank}[u_1] = 1$	$\{u_1\}$ l.i.	$\{u_1\}$
$\text{rank}[u_1 u_2] = 1$	$\{u_1, u_2\}$ l.d.	$\{u_1\}$
$\text{rank}[u_1 u_2 u_3] = 2$	$\{u_1, u_3\}$ l.i	$\{u_1, u_3\}$
$\text{rank}[u_1 u_2 u_3 u_4] = 2$	$\{u_1, u_3, u_4\}$ l.d.	$\{u_1, u_3\}$
$\text{rank } B = 3$	$\{u_1, u_3, u_5\}$ l.i	$\{u_1, u_3, u_5\}$

$\therefore$ $\{u_1, u_3, u_5\}$ is a basis of W ($= \text{col}(B)$) which is a subset of

Thm 2 ... (4) 1) If B' is any RE form of a matrix B and the leading ones occur in columns $j_1, \dots, j_r$ of B', then the columns $j_1, \dots, j_r$ of B are a basis of $\mathrm{col}(B)$

P. 205

$\therefore$ 2) $\operatorname{rank} B = \dim \mathrm{col}(B)$ $\left(= \dim \mathrm{row}(B)!\right)$

Corollary: AMAZING FACT: for any matrix A, $\dim \mathrm{row}(A) = \operatorname{rank} A = \dim \mathrm{col}(A)$!

L19

Problem III Extending bases (usually of a subspace to one for $\mathbb{R}^p$).

e.g. Given $W = \mathrm{span}\{(0,0,1,0), (0,1,1,0)\}$. $\dim W = 2$

$\{(0,0,1,0), (0,1,1,0)\}$ basis.

Extend the basis $\{\underset{v_1}{(0,0,1,0)}, \underset{v_2}{(0,1,1,0)}\}$ to a basis of $\mathbb{R}^4$.

Write $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ & v_3 & & \\ & v_4 & & \end{bmatrix}$; want to choose v_3, v_4 so that $\mathrm{row}(A) = \mathbb{R}^4$ i.e. $\operatorname{rank} A = 4$

$\sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ & v_3 & & \\ & v_4 & & \end{bmatrix}$ need leading ones in col 1 & 4

$\therefore$ set $v_3 = (1, 0, 0, 0)$
$v_4 = (0, 0, 0, 1)$

$\sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A'$; $\operatorname{rank} A' = 4$

$\therefore$

$\{v_1, v_2, \underline{v_3, v_4}\}$ is a basis of $\mathbb{R}^4$

Summary on subspaces associated to an $m \times n$ matrix A

1) $\text{row}(A)$ is a s.s. of $\mathbb{R}^n$; $\dim \text{row}(A) = \text{rank} A$

2. $\text{col}(A)$ ss of $\mathbb{R}^m$; $\text{col}(A) = \underbrace{\{Ax \mid x \in \mathbb{R}^n\}}_{=: \text{im}(A)}$; $\dim \text{col}(A) = \text{rank} A$

3. $\ker(A)$ is a s.s. of $\mathbb{R}^n$; $\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$

$$\dim \ker A = n - \text{rank} A$$

or
$$\dim \ker A + \text{rank} A = n$$

Thm 6 p. 209

$$\left.\begin{aligned} \dim \ker A + \dim \text{col}(A) &= n \\ \dim \ker A + \dim \text{im}(A) &= n \\ * \quad \dim\{x \mid Ax = 0\} + \dim\{Ax \mid x \in \mathbb{R}^n\} &= n \end{aligned}\right\}$$

"conservation of dimension"

A $m \times n$

$$\underset{\in \mathbb{R}^n}{x} \longmapsto \underset{\mathbb{R}^m}{Ax \in}$$

Remark. $\ker A = \{x \mid Ax = 0\}$; $A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$

$$= \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} x = 0 \right\}$$

$$= \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$= \{ x \in \mathbb{R}^n \mid x$ is orthogonal to all the rows of $A \}$

$\dim \ker A + \dim \text{row}(A) = n$

e.g. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix}$ $\quad$ $\left.\begin{array}{l} row(A) \subseteq \mathbb{R}^3 \\ col(A) \subseteq \mathbb{R}^3 \end{array}\right\}$ not same!

$\ker A \subseteq \mathbb{R}^3$

$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & +1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ $x = -s$, $y = s$, $z = s$

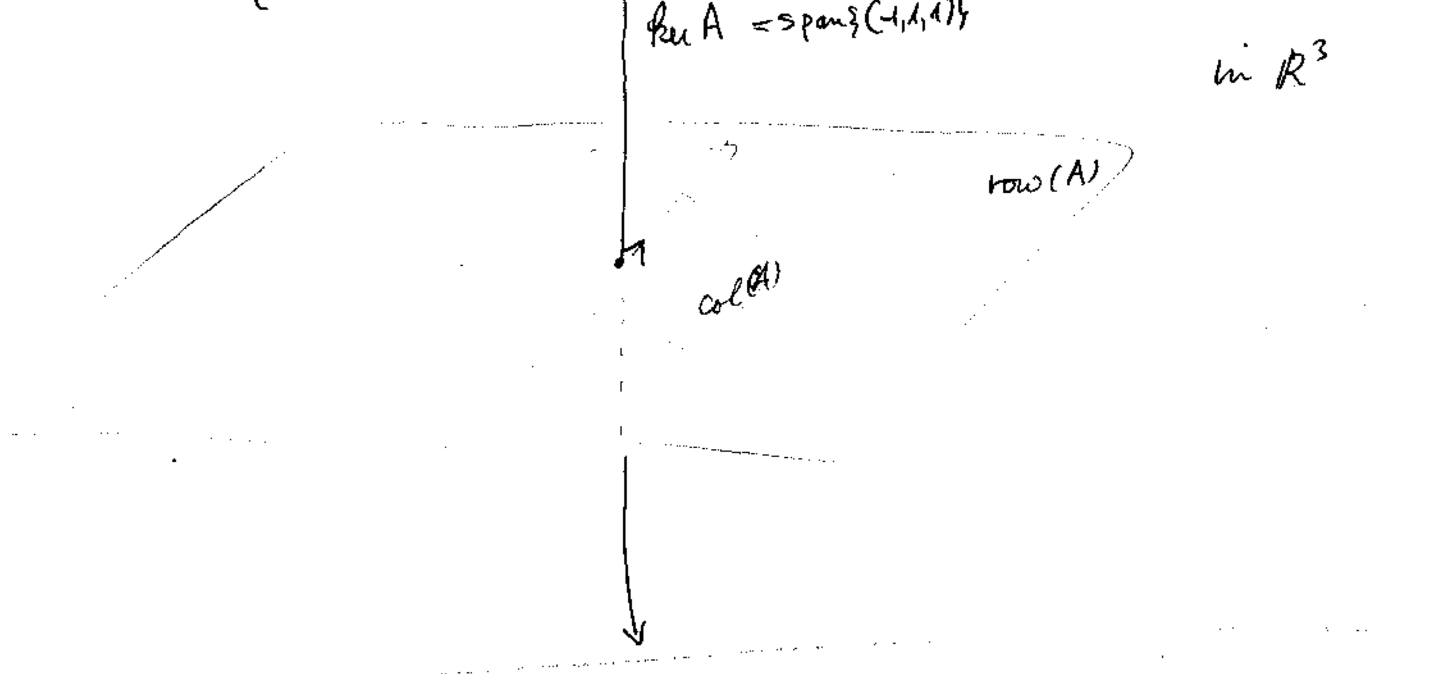
col A: $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 0 & -2 \end{vmatrix}$ $(-2, +1, 1)$

basis of

row(A) is $\{(1,1,0), (0,1,-1)\}$

col(A) is $\{(1,1,-1), (1,0,-2)\}$

ker A is $\{(-1,1,1)\}$



extend basis of col A to basis of $\mathbb{R}^3$: note $[e_1 e_2 e_3]$

$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, which has rank 3 $\quad \therefore \{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$

ker A to basis of $\mathbb{R}^3$: $[-1 \ 1 \ 1] = [v]$

4.5 Orthogonality   4.5.1 - done

4.5.2 Orthogonal sets and projections

- Def^n $\{u_1, \dots, u_k\} \subset \mathbb{R}^n$ is **orthogonal** if

  ① $u_i \cdot u_j = 0 \quad i \neq j$

  ② $u_i \neq 0 \quad , \forall i \quad (\Leftrightarrow u_i \cdot u_i \neq 0)$

(if in addition $\|u_i\| = \sqrt{u_i \cdot u_i} = 1$, also called orthonormal)

ex. (Thm 4) If $\{u_1, \dots, u_k\}$ is orthogonal, $\|u_1 + u_2 + \dots + u_k\|^2 = \|u_1\|^2 + \dots + \|u_k\|^2$

For ↓ Thm 5 p. 215

If $\{u_1, \dots, u_k\}$ is orthogonal, it is l.i.

Pf Suppose $c_1u_1 + \dots + c_ku_k = 0$ for some scalars $c_1, \dots, c_k$.
Take dot product of both sides with u_i to obtain

$$c_i(u_i \cdot u_i) = 0 \Leftrightarrow c_i = 0 \quad , i = 1, \dots, n. \quad \blacksquare$$

Cor $\{u_1, \dots, u_n\} \subset \mathbb{R}^n$ is orthogonal, it is a basis of $\mathbb{R}^n$!

Thm 6 (P216) If W is a subspace of $\mathbb{R}^n$ and $\{u_1, \dots, u_k\}$ is an orthogonal basis of W, for any $w \in W$ we have

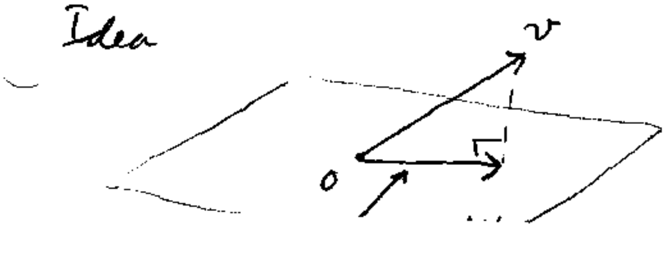
$$w = \left(\frac{w \cdot u_1}{\|u_1\|^2}\right) u_1 + \left(\frac{w \cdot u_2}{\|u_2\|^2}\right) u_2 + \dots + \left(\frac{w \cdot u_k}{\|u_k\|^2}\right) u_k$$

"Fourier" coefficients

Pf. We know $w = c_1u_1 + \dots + c_ku_k$ for some scalars.
Take dot product on both sides with u_i to obtain

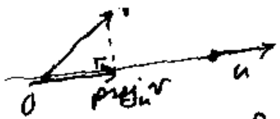
$$w \cdot u_i = c_i(u_i \cdot u_i) \quad \text{so} \quad c_i = \frac{w \cdot u_i}{\|u_i\|^2} \quad \blacksquare$$

Idea



W - "interesting" subspace
Given $v \in \mathbb{R}^n$, find vector in W closest to v: the best

eg $W = \text{span}\{u\}$



$\text{proj}_W v = \left(\frac{v \cdot u}{\|u\|^2}\right) u.$

Def^n p223 (4.6.2) If $\{w_1, \ldots, w_k\}$ is an orthogonal basis of a subspace W, and $v \in \mathbb{R}^n$, then

$$\text{proj}_W v = \left(\frac{v \cdot w_1}{\|w_1\|^2}\right) w_1 + \cdots + \left(\frac{v \cdot w_k}{\|w_k\|^2}\right) w_k$$

is the <u>(orthogonal) projection of v on W</u> ("proj(v,W)" in book)

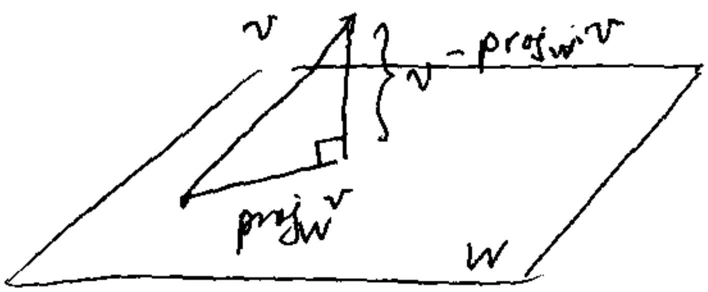
e.g. $W = \{(x, y, z) \in \mathbb{R}^3 \mid x - z = 0\}$

$\{\underset{w_1}{(1,0,1)}, \underset{w_2}{(0,1,0)}\}$ is an orthogonal basis of W;

$$\therefore \text{proj}_W(x,y,z) = \left(\frac{(x,y,z)\cdot w_1}{\|w_1\|^2}\right) w_1 + \left(\frac{(x,y,z)\cdot w_2}{\|w_2\|^2}\right) w_2$$

$$= \left(\frac{x+z}{2}\right)(1,0,1) + y\,(0,1,0)$$

$$= \left(\frac{x+z}{2}, y, \frac{x+z}{2}\right)$$



<u>Approximation Thm</u> (2, 4) 225-227 : 1) $v - \text{proj}_W v$ is $\perp$ to every vector in W

2) $\text{proj}_W v$ is the best approximation to v by vectors in W: i.e. $\forall w \in W$

$$\| \quad \cdots \quad v \| \leq \|v - w\|$$

Pf: (1) calc^n.
(2) $v - w = (v - \text{proj}_W v) + (\text{proj}_W v - w)$;

e.g. $W = \{ (x,y,z) \mid x - z = 0 \}$

The best approximation to $(1,0,0)$ in W is

$$\text{proj}_W (1,0,0) = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right).$$

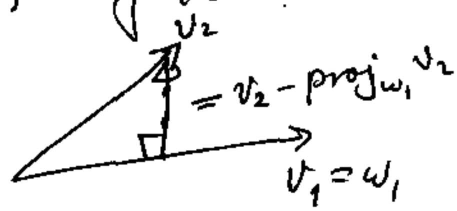
— How can we find orthogonal bases for a subspace W?

4.5.3 Gram-Schmidt Orthogonalization algorithm

(P.216) idea: convert any basis of W into an orthogonal one.

Let $\{v_1, \ldots, v_k\}$ be a basis for a subspace W of $\mathbb{R}^n$

We produce an orthogonal basis $\{w_1, \ldots, w_k\}$ as follows:



Set

- $w_1 = v_1$

- $$w_2 = v_2 - \text{proj}_{w_1} v_2 = v_2 - \left(\frac{v_2 \cdot w_1}{\|w_1\|^2} \right) w_1$$

- $$w_3 = v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3 = v_3 - \left(\frac{v_3 \cdot w_1}{\|w_1\|^2} \right) w_1 - \left(\frac{v_3 \cdot w_2}{\|w_2\|^2} \right) w_2$$

e.g.

$$W = \{ (x,y,z,w) \in \mathbb{R}^4 \;/\; x+y+z+w = 0 \}$$

(1) Find a basis:

$$\begin{array}{cccc} & r & s & t \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

$x_1 = -r-s-t$
$x_2 = r$
$x_3 = s$
$x_4 = t$

$$\{ \underset{v_1}{(-1,1,0,0)}, \underset{v_2}{(-1,0,1,0)}, \underset{v_3}{(-1,0,0,1)} \}$$

G-S $\quad w_1 = (-1,1,0,0)$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1 = (-1,0,1,0) - \frac{1}{2}(-1,1,0,0)$$

$$= \left(-\tfrac{1}{2}, \tfrac{1}{2}, 1, 0\right) \qquad \|w_2\|^2 = \tfrac{3}{2}$$

$$w_3 = v_3 - \left(\frac{v_3 \cdot w_1}{\|w_1\|^2}\right) w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2$$

$$= (-1,0,0,1) - \frac{1}{2}(-1,1,0,0) - \frac{\frac{1}{2}}{\frac{3}{2}} \tfrac{1}{3}\left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0\right)$$

$-\frac{1}{2} + \frac{1}{6}$

$$= \left(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 1\right)$$

$$\therefore \{ \underset{w_1}{(-1,1,0,0)}, \underset{w_2}{\left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0\right)}, \underset{w_3}{\left(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 1\right)} \}$$

is an orthog basis of W.

$1+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}$

$\frac{4}{3}$

ex. Find least appox to $u = (1,0,0,0)$ in W.

lect 18

(69,70)

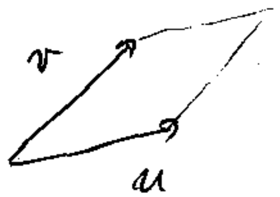
7

Ch 2 Determinants : a number associated to a matrix A

Geometrically – are about area, volume, hypervolume

Algebraically – tell us when A is invertible.

recall :



Area = $\| u \times v \|$

$u = (a, b, 0)$ $\quad u \times v = (0, 0, ad - bc)$

$v = (c, d, 0)$

$\therefore$ area = $|ad - bc|$

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Defn The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

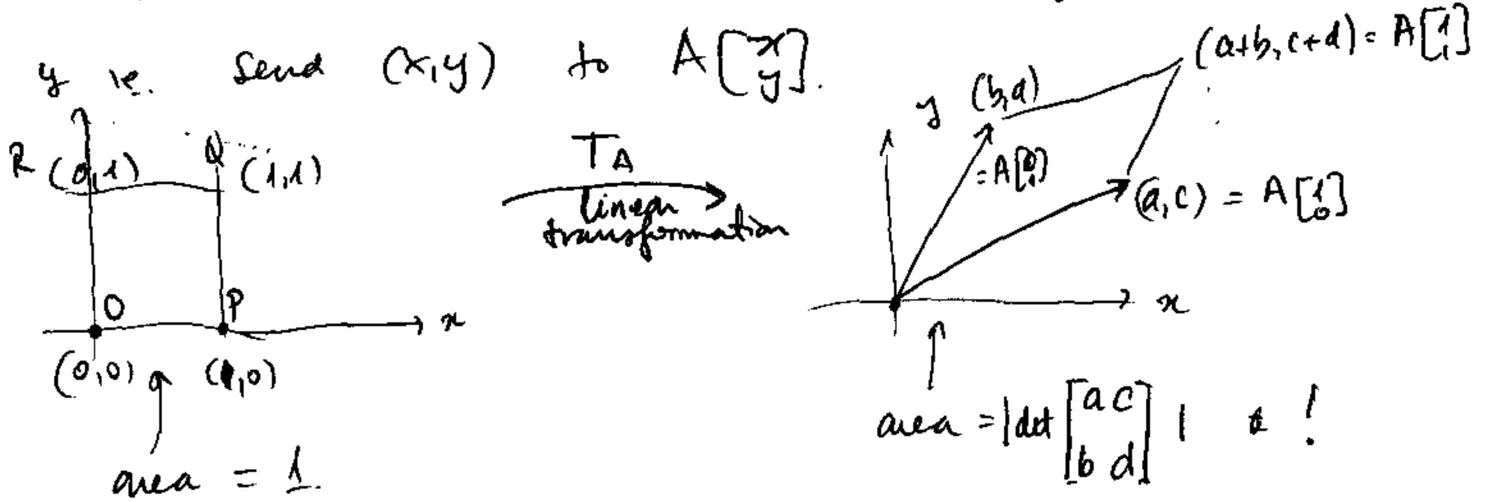
Write
$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; (note: $\det A = \det A^t$!)

01 W/100   Lect. 20, W2004

Determinants are related to area in another way:

Suppose we transform the unit square by multiplying each point in the plane by a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

i.e. send (x,y) to $A\begin{bmatrix} x \\ y \end{bmatrix}$.



Exercise ① Show that area $T_A(\square) = |\det. A|$ area $\square$

In $\mathbb{R}^3$ the vol. of parallelepiped  (7.2)

is $|u \cdot v \times w|$. (see 8.13 P 288) 72

If $u = (a_1, b_1, c)$, $v = (a_2, b_2, c_2)$, $w = (a_3, b_3, c_3)$

then $u \cdot v \times w = (a_1, b_1, c_1) \cdot \underbrace{\left(\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \right)}_{v \times w\,!}$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

(8.3)

Def^n the determinant of $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ Sarrus

(see P. 278-279)

3x3 det

(Computed, defined in terms of 2x2!)

26/3/02

Def^n Suppose A is an $n \times n$ matrix $= [a_{ij}]$ and we know how to compute $(n-1)\times(n-1)$ determinants. Then

(Thm 8.8)

$$\det A = a_{11} C_{11}(A) + a_{12} C_{12}(A) + \cdots + a_{1n} C_{1n}(A)$$

where $C_{ij}(A) = (-1)^{i+j} \det$ ($(n-1)\times(n-1)$ matrix obtained by deleting ith row & jth column of A)

A_{ij} mtx

"(i,j) cofactor"

pattern →

+	−	+	−
−	+	−	
+	−	+	−
−			

Properties of determinants – what operations don't change, then which do ; Goal: how to evaluate them quickly

pblm: count # of multn

2×2 – 2 mult of 2 numbers
3×3 – $6 = 3\cdot 2$ " 3 "
4×4 – $4\times 3\times 2$ " 4 "
25×25 – $25!$ " 25

Now $25! \approx 1.6\times 10^{25}$; so for a 25×25 matrix, there are $\sim 4\times 10^{26}$ m
On a 1 THz machine @ 1 mult / clockcycle

$$\frac{4\times 10^{26}}{10^{12}\times \pi\times 10^{7}} \text{ years} > 10^{7} \text{ years.}$$

- main strategies

- expand along any row or column – get same answer (pick row, col with many zeros)

effect of row ops:
- adding multiple of one row (resp. column) to another row (resp col) doesn't change det!
- interchanging rows (resp. cols) changes sign
- mult row by k changes det by k.

- upper (lower) $\triangle$ular matrices ; determinants are easy to compute

$$\det\begin{pmatrix} a_{11} & * & * \\ 0 & a_{22} & * \\ 0 & 0 & a_{33}\end{pmatrix} = a_{11}\cdot a_{22}\cdot a_{33}\ !$$

e.g.

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 3\end{vmatrix} \overset{\text{def}^n}{=} 1\, C_{11}(A) + 0\, C_{12} + 1\, C_{13}(A)$$

$$= 1\begin{vmatrix} 6 & 2 \\ -1 & 3\end{vmatrix} + 0(-1)\begin{vmatrix} 2 & 2 \\ 1 & 3\end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 6 \\ 1 & -1\end{vmatrix}$$

• another row

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 3 \end{vmatrix} \overset{\text{row } 3}{=} 1 \cdot \begin{vmatrix} 0 & 1 \\ 6 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}$$

$$= -6 + 18 = 12!$$

• a column

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 3 \end{vmatrix} \overset{\text{col } 2}{=} 0 \cdot - \begin{vmatrix} * \end{vmatrix} + 6 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 12!$$

ex.: col 1 : $\begin{vmatrix} 6 & 2 \\ -1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 6 & 2 \end{vmatrix} = 20 - \; -6 = 12$

N.B. This means $\det A = \det A^t$!

eg Upper Δlar

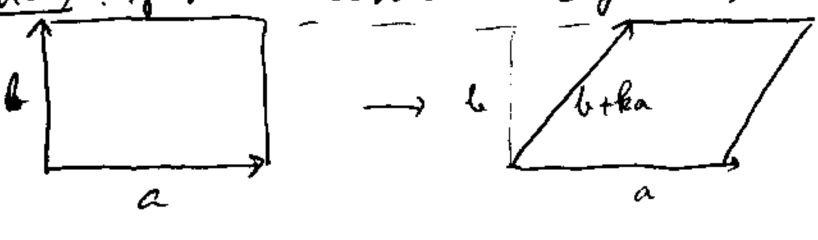
$$\begin{vmatrix} 7 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} \overset{\text{col } 1!}{=} 7 \cdot \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} + 0 + 0$$

$$= 7 \cdot 4 \cdot 6 = \text{product of diag elts.}$$

Thm 4 P. 75 (1) det A can be expanded along any row or column. (2) If A is upper (or lower) Δular, det A is the product of the diagonal entries.

Corollary: If A has a row or col. of zeros, $\det A = 0$.

Notes:



area same.

Effect of row operations on det A:

- $12 = |A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 3 \end{vmatrix}$ $\quad -2R_1+R_2 \to R_2 \quad$ $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & -1 & 3 \end{vmatrix} \overset{\text{row 1}}{=} \begin{vmatrix} 6 & 0 \\ -1 & 3 \end{vmatrix} - 0 + \begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}$

$= 18 - 6 = 12$

$-R_1+R_3 \to R_3 \quad \begin{vmatrix} 1 & 0 & 1 \\ 0 & 6 & 0 \\ 0 & -1 & 2 \end{vmatrix} \overset{\text{col 1}}{=} \begin{vmatrix} 6 & 0 \\ -1 & 2 \end{vmatrix} = 12$!

- $\left(\frac{1}{6}\right)R_2 \to R_2 \quad \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} \overset{\text{col 1}}{=} \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2 = 12 \cdot \left(\frac{1}{6}\right)$

(don't need this)
- $R_1 \leftrightarrow R_3 \quad \begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} 0 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \overset{\text{col 1}}{=} \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = -2$ (change of sign)

Thm 2 P77 (a) adding a multiple of a row to a different row does not change det A

(b) multiply a single row of A by k changes det A by the factor k

(c) interchanging 2 rows of A changes the sign of det A.

Rmk: same statements hold for "column" replacing "row".

Cor. ① if A has a rows which is a multiple of another, det A = 0 eg $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix} = 0$

② $\det(kA) = k^n \det A$ if A is $n \times n$, $k \in \mathbb{R}$.

Remark $A \sim A'$ (RE form: upper Δular!)

If we use j row interchanges, and multiply various rows by $k_1, \dots, k_r$, then $\det A' = (-1)^j k_1 \cdots k_r \det A$ $\quad (k_i \neq 0,\ i=1,\dots,r)$

Strategy for computing $\det A$

1. Use row/col ops to (almost) clean a chosen row or col
2. expand along chosen col/row
3. return to 1 if necessary

eg

$$\begin{vmatrix} 1 & -2 & 3 & -1 \\ 1 & 1 & -2 & 0 \\ 2 & 0 & 4 & -5 \\ 1 & 4 & 4 & -6 \end{vmatrix} \underset{\text{(clean row 2)}}{\overset{\alpha C_1 + C_j \to C_j}{=}} \begin{vmatrix} 1 & -3 & 5 & -1 \\ 1 & 0 & 0 & 0 \\ 2 & -2 & 8 & -5 \\ 1 & 3 & 6 & -6 \end{vmatrix} \overset{\text{row } 2}{=} - \begin{vmatrix} -3 & 5 & -1 \\ -2 & 8 & -5 \\ 3 & 6 & -6 \end{vmatrix}$$

$$\overset{C_j + C_3 \to C_j}{=} - \begin{vmatrix} 0 & 0 & -1 \\ 13 & -17 & -5 \\ 21 & -24 & -6 \end{vmatrix} = (-)(-) \begin{vmatrix} 13 & -17 \\ 21 & -24 \end{vmatrix} = 3 \cdot \begin{vmatrix} 13 & -17 \\ 7 & -8 \end{vmatrix} = 3 \begin{vmatrix} -1 & -1 \\ 7 & -8 \end{vmatrix} = 3 \cdot (15) = 45$$

$-2R_2 + R_1 \to R_1$

e.g. (complicated)
λ unknown

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \overset{-R_1 + R_3 \to R_3}{=} \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 1+\lambda & 0 & -1-\lambda \end{vmatrix} \overset{C_1 + C_3 \to C_3}{=} \begin{vmatrix} 3-\lambda & 2 & 7-\lambda \\ 2 & -\lambda & 4 \\ 1+\lambda & 0 & 0 \end{vmatrix}$$

$$= (1+\lambda) \begin{vmatrix} 2 & 7-\lambda \\ -\lambda & 4 \end{vmatrix} = (1+\lambda)\{8 + \lambda(7-\lambda)\} = (1+\lambda)(-\lambda^2 + 7\lambda + 8)$$

$$= (1+\lambda)(8-\lambda)(1+\lambda)$$

$$= (1+\lambda)^2(8-\lambda).$$

Rmk

I

$$\begin{vmatrix} a & b & c & 0 & 0 \\ d & e & f & b & e \\ g & h & k & c & d \end{vmatrix}$$

✓

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} ; \det = -1.$$

×

II

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -10 & 1 & 5 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 7 ; \quad -2R_2 + 3R_3 \to R_2 \quad \begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 3 \\ 2 & 0 & 1 \end{vmatrix} = 2 \cdot (-7) = -14$$

Determinants and Matrix Inverses

Thm 2: (P.82) An $n\times n$ matrix A is invertible $\Leftrightarrow \det A \neq 0$. (add to list...)

Pf "$\Rightarrow$". Assume A invertible, then $A \sim I_n$, i.e. RC form of A is I_n

$\therefore$ from above $\underbrace{\det A = (-1)^j \frac{1}{k_1}\cdots\frac{1}{k_n}}_{} \cdot \det(I_n) \neq 0$.

"$\Leftarrow$" If $\det A \neq 0$, $\det A = (-1)^j \frac{1}{k_1}\cdots\frac{1}{k_r}\det(\tilde{A})$ ($\tilde{A}$ RC form of A)

implies $\det\tilde{A}\neq 0$; since $\tilde{A}$ is upper $\triangle$ular, $\Rightarrow \tilde{A}$ has no row of zeros,

So rank $A = n$; $\therefore A$ is invertible

(L22) Thm 1 (P. 82) $\det(AB) = \det A \det B$ * (See web!)

Different pf from book: If A is not invertible, then neither is AB (remember?)

so both sides of eqn are zero.

Suppose A is invertible. Consider $[A|AB]$; now $A \sim I_n$; $\cdots$?

$\sim [I_n|M]$; claim: $M = B$! We know row reduction doesn't change

ker; so $\ker[A|AB] = \ker[I|M]$; Note: $\{e_1,\ldots,e_n\}$ std basis of $\mathbb{R}^n$;

$$\begin{array}{cc} & \begin{array}{cc} 2n & 1 \end{array} \\ n & [A|AB]\begin{bmatrix} Be_i \\ -e_i\end{bmatrix}_{2n} \end{array} = ABe_i - ABe_i = 0;\ i=1,\ldots,n.$$

Hence, $0 = [I|M]\begin{bmatrix} Be_i \\ -e_i\end{bmatrix} = Be_i - Me_i = 0$; $i=1,\ldots,n$ $\therefore M = B$.

$\therefore [A|AB] \sim [I_n|B]$; $\det A = (-1)^j \frac{1}{k}\cdots\frac{1}{k}\det I_n$; same row ops take

Chp 2.3 Evals, Evecs, Diagonalization (Square matrices) ®

- Many processes modelled by $\underset{\text{input}}{x} \mapsto \underset{\substack{\text{output}\\\text{input}}}{Ax} \mapsto A^2x \mapsto \dots \quad \underset{N\gg 1}{A^N x}$?

- if $A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$, $A^N = \begin{bmatrix} a_{11}^N & & 0 \\ & \ddots & \\ 0 & & a_{nn}^N \end{bmatrix}$ easy $|a_{ii}|$ largest dets behaviour.

- not all matrices are diagonal: what are the special properties of diagonal matrices? Goal: "diagonalize" matrices where possible

eg $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad D\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad D\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$De_1 = e_1 \qquad De_2 = 4e_2 \qquad \{e_1, e_2\}$ basis $\mathbb{R}^2$

eg $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ $\quad A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 4\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$Av_1 = v_1 \qquad Av_2 = 4v_2$

$$\underset{\text{cols}}{A}[v_1\ v_2] = [Av_1\ Av_2] = [v_1\ 4v_2] = [v_1\ v_2]\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}_{D}$$

note. $P = [v_1\ v_2] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ inv (so $\{v_1, v_2\}$ basis of $\mathbb{R}^2$) so

$$AP = PD \quad \text{or} \quad D = P^{-1}AP$$
$$A = PDP^{-1}$$

$$A^N = PD\underbrace{P^{-1}\,P}DP^{-1} \dots \underbrace{\ }PDP^{-1} = PD^NP^{-1} \text{ easy!}$$

Def^n P.90 An $n \times n$ matrix is diagonalizable ("diag'ble") if there is an invertible matrix P and a diagonal matrix D s.t.

$$P^{-1}AP = D$$

eg. D, A above are diag'ble (every diagonal matrix is diag'ble $P = I$)

[illegible]

How can we (try to) diagonalize any matrix A?

Need special vectors v_1, v_2 above:

Defn P.90 • A non-zero vector v is an eigen vector of a matrix A if $Av = \lambda v$ for some scalar λ, called an eigenvalue of A (corresp. to evec (say v "corresponds to" eval λ).

eg. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$, 1 is an eval $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ an evec corres. to $\lambda = 1$

4 " $v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ " $\lambda = 4$

eg $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ 1, 4 are evals; e_1, e_2 are corres evecs. Same evals, differen evecs

• NB in our example above, cols of $P = [v_1\ v_2]$ were evecs of A

Thm 3 (P.94) A is diag'ble $\iff$ there is a basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ consisting of evecs of A. Moreover, if it is, if $P = [v_1 \cdots v_n]$ (v_j = j^{th} col) then $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is diagonal, where $\lambda_1, \dots, \lambda_n$ are the evals of A.

[$D = P^{-1}AP \iff AP = PD$ (P inv) $\iff A[v_1 \cdots v_n] = [v_1 \cdots v_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix}$ & $\{v_1, \dots, v_n\}$ basis of $\mathbb{R}^n$

$\iff [Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n]$ & $\{v_1, \dots, v_n\}$ basis of $\mathbb{R}^n$

$\iff$ $Av_1 = \lambda v_1$, $\vdots$, $Av_n = \lambda_n v_n$, $\{v_1, \dots, v_n\}$ basis of $\mathbb{R}^n$]

How can we find enough evecs? evals?

• λ is an eval of A $\iff \exists\, v \in \mathbb{R}^n,\ v \neq 0$ st $Av = \lambda v$

$\iff \exists\, v \neq 0$ st $(A - \lambda I)v = 0$

$\iff (A - \lambda I)$ is not inv

$\iff \det(A - \lambda I) = 0$!

Thm 2 (P 91) λ is an eval of A iff $\det(A - \lambda I) = 0$.

e.g. $\begin{vmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$

$\lambda = 1, 4$ are evals.

Find evecs: solve $Av = \lambda v$ for $\lambda = 1$, then $\lambda = 4$!

$(A - \lambda I)v = 0$ "

i.e. solve $(A - I)v = 0$ & $(A - 4I)v = 0$

<u>Rmk</u>. λ an eval of A, $E_\lambda := \{ v \in \mathbb{R}^n \mid Av = \lambda v \}$

$= \{ v \in \mathbb{R}^n \mid (A - \lambda I)v = 0 \} = \ker(A - \lambda I)$!

is the <u>eigenspace</u> corresponding to eval λ;

$E_1 = \text{Span}\{(1,2)\}$, $E_4 = \text{Span}\{(-1,1)\}$ IN THIS CASE.

e.g. $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad c_J(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \iff \lambda = \pm i$

J has no <u>real</u> evals. Can diagonalize J but need to use complex vectors ... (Say J <u>not</u> diag'ble over <u>reals</u>)

e.g. $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad c_B(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2$;

only eval is $\lambda = 2$; (<u>Rmk</u>. evals of triangular (upper or lower) matrices are the diagonal entries!)

<u>Find evecs</u>:

$E_2 = \{ v \mid (B - 2I)v = 0 \} = \ker(B - 2I)$; $\quad [B - 2I \mid 0] = \left[\begin{array}{cc|c} \overset{s}{0} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \begin{array}{l} x = s \\ y = 0 \end{array}$

$E_2 = \text{Span}\{(1,0)\}$. There isn't a basis of $\mathbb{R}^2$ consisting of evecs for B!

<u>B</u> is not diag'ble.

e.g. $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ $\quad C_A(\lambda) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 1+\lambda & 0 & -1-\lambda \end{vmatrix}$

$= \begin{vmatrix} 3-\lambda & 2 & 7-\lambda \\ 2 & -\lambda & 4 \\ 1+\lambda & 0 & 0 \end{vmatrix} \overset{\text{row 3}}{=} (1+\lambda) \begin{vmatrix} 2 & 7-\lambda \\ -\lambda & 4 \end{vmatrix} = \cdots = (1+\lambda)^2(8-\lambda)$ $\quad$ N.B. <u>Need to factorize</u>

Find evecs: $E_{-1} = \ker(A+I)$; $[A+I|0] = \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$ (81)

$$\sim \cdots \left[\begin{array}{ccc|} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x = -\frac{s}{2} - t \\ y = s \\ z = t \end{array} \quad ; t \in \mathbb{R} \quad \therefore \{ \overset{v_1}{(-\tfrac{1}{2}, 1, 0)}, \overset{v_2}{(-1, 0, 1)} \} \text{ basis for}$$

E_{-1} (2 l.i. evecs so far!)

$E_8 = \{v \mid Av = 8v\} = \ker(A - 8I)$; $[A - 8I|0] = \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$

$$\sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x = s \\ y = +\frac{1}{2}s \\ z = s \end{array} \quad s \in \mathbb{R} \qquad \therefore \{ \overset{v_3}{(1, \tfrac{1}{2}, 1)} \} \text{ basis for } E_8$$

is $\{v_1, v_2, v_3\}$ basis of $\mathbb{R}^3$, ? Set $P = [v_1\ v_2\ v_3]$ v_j = cols

$$= \begin{bmatrix} -\frac{1}{2} & -1 & 1 \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix} ; \det P = \begin{vmatrix} -\frac{1}{2} & -1 & 2 \\ 1 & 0 & \frac{1}{2} \\ \cancel{0} & \cancel{1} & 0 \end{vmatrix} = - \begin{vmatrix} -\frac{1}{2} & 2 \\ 1 & \frac{1}{2} \end{vmatrix} = -\left(-\tfrac{1}{4} - 2\right) = 2\tfrac{1}{4} \neq 0$$

So $AP = A[v_1\ v_2\ v_3] = [Av_1\ \ Av_2\ \ Av_3] = [-v_1\ \ -v_2\ \ 8v_3]$

$$= [v_1\ \ v_2\ \ v_3] \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad \text{i.e.} \quad AP = PD \quad \text{so} \quad D = P^{-1}AP$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Remark ① (Thm 9, P241) Eigenvectors corresp. to distinct evals are always l.i.

① $\Longrightarrow$ ② (Thm 4, P95) An $n \times n$ matrix with n distinct evals is always diag'ble (each different eval yields l.i. evec) • $\begin{bmatrix} 1 & 6 & 8 \\ 0 & 0 & \pi \\ 0 & 0 & 40 \end{bmatrix}$ diag'b
• ("$\Leftarrow$" is not true)

① $\Longrightarrow$ ③ If $\lambda_1, \ldots, \lambda_k$ are the distinct evals of A, then A is diag'ble $\Longleftrightarrow$ $\dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_k} = n$

Diagonalization Algorithm (p. 96)

(1) compute $C_A(\lambda)$ and so find distinct evals of A
(2) find a basis for each eigenspace
(3) If there are n vectors in all, A is diag'ble; if not it isn't
(4) If A is diag'ble, let $P = [v_1 \dots v_n]$ be constructed with vectors from (2) as columns; let $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$ have diagonal entries corresponding to the evals assoc to $v_1, \dots v_n$ ie if $Av_i = \lambda_i v_i$, then $D = \begin{bmatrix} \ddots & & 0 \\ & \lambda_i & \\ 0 & & \ddots \end{bmatrix}$

4.9 Linear Transformations ($\mathbb{R}^n \to \mathbb{R}^m$)

Recall example $W = \{(x,y,z) \mid x - z = 0\}$; had orthogonal projection

$v = (x,y,z)$

$$P = \text{proj}_W(x,y,z) = \tfrac{1}{2}(x+z,\ 2y,\ x+z) \quad ; \quad P(v) = \dots$$

$P = \text{proj}_W : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ (domain $\mathbb{R}^3$, range is $\mathbb{R}^3$ (or W!))

P has special properties; let $v = (x,y,z)$, $w = (x',y',z')$

1) $P(v+w) = P(x+x', y+y', z+z') = \tfrac{1}{2}(x+x'+z+z',\ 2(y+y'),\ x+x'+z+z')$
$= \tfrac{1}{2}(x+z,\ 2y,\ x+z) + \tfrac{1}{2}(x'+z',\ 2y',\ x'+z')$
$= P(v) + P(w)$!

2) if $k \in \mathbb{R}$, $P(kv) = P(kx, ky, kz) = \tfrac{1}{2}(kx+kz,\ 2ky,\ kx+kz)$
$= k\left(\tfrac{1}{2}(x+z,\ 2y,\ x+z)\right)$
$= kP(v)$.

• another example: Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$, define a function

$T_A : \mathbb{R}^3 \to \mathbb{R}^2$ by $T_A(v) = Av$ for $v \in \mathbb{R}^3$ (col.)

Note: If $v, w \in \mathbb{R}^3$ & $k \in \mathbb{R}$

① $T_A(v+w) = A(v+w) = Av + Aw = T_A(v) + T_A(w)$

② $T_A(kv) = A(kv) = k.(Av) = k.T_A(v)$

Def^n If U, V are vector spaces a function $T: U \to V$ is a linear transformation

if ① $T(u+u') = T(u) + T(u')$ $\forall u, u' \in U$

② $T(ku) = k.T(u)$ $\forall u \in U, k \in \mathbb{R}$.

e.g. $P = \text{proj}_W$, $T = T_A$ are l.t.

eg $F: \mathbb{R}^2 \to \mathbb{R}$ def'd by $F(x,y) = xy$ is not,

since $F(1,0) = 0$, $F(0,1) = 0$ but
$F((1,0) + (0,1)) = F(1,1) = 1 \neq F(1,0) + F(0,1)$

Rmk: They are called "linear" because if $S: \mathbb{R}^3 \to \mathbb{R}^3$ is a l.t. $\Leftrightarrow$ S lends through 0 to lines through 0.
(and other lines to other lines)

e.g. $P = \text{proj}_W$ as before, $L = \{(0, s, 2s) \mid s \in \mathbb{R}\}$ = line through 0, dir^n $(0,1,2)$

$P(L) = \{ \frac{1}{2}(2s, 2s, 2s) \mid s \in \mathbb{R}\} = \{(s,s,s) \mid s \in \mathbb{R}\}$ = line through 0, dir^n $(1,1,1)$.

Thm 1 If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a l.t., it is completely determined
P208 by what it does to a basis of $\mathbb{R}^n$! In fact if $\{v_1, \dots, v_n\}$ is any basis of $\mathbb{R}^n$ and $\{w_1, \dots, w_n\}$ are any vectors in $\mathbb{R}^m$, there is a unique l.t. $S: \mathbb{R}^n \to \mathbb{R}^m$ st $S(v_i) = w_i$, $i = 1, \dots, n$.

T_A wasn't so special!

Thm 2: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a l.t. Then there is an $m \times n$ matrix A s.t. $T(v) = Av$ for all $v \in \mathbb{R}^n$!
Moreover, if $\{e_1, \ldots, e_n\}$ is the standard ordered basis of $\mathbb{R}^n$, $A = [T(e_1), \ldots, T(e_n)]$ is the matrix. (Standard matrix of T)

Pf. Let $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$; & $A = [T(e_1) \cdots T(e_n)]$ cols

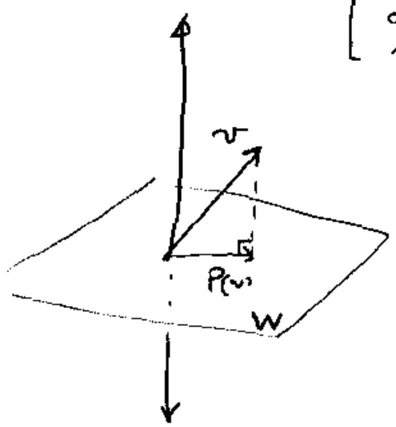
$$\begin{aligned} T(v) &= T(x_1 e_1 + \cdots x_n e_n) \\ &= x_1 T(e_1) + \cdots + x_n T(e_n) \\ &= [T(e_1) \cdots T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= Av! \end{aligned}$$

---

e.g. $P = \text{proj}_W$; Let's find its matrix:

$P(x,y,z) = \frac{1}{2}(x+z, 2y, x+z)$; $P(e_1) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$ $P(e_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $P(e_3) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$

$\therefore P(x,y,z) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{bmatrix} x+z \\ 2y \\ x+z \end{bmatrix}$ ✓



P sends some vectors to zero (normal line)

P sends all vectors onto W; "fills out" W.

Def^n (P315) Let $T: U \to V$ be a l.t. Then

the kernel of T = $\ker T = \{u \in U \mid T(u) = 0\}$ — s.s. of U

& the image of T = $\text{im}\, T = \{T(u) \in V \mid u \in U\}$ — s.s. of V

eg $P(v) = \frac{1}{2}(x+z,\ 2y,\ x+z)$; $\quad P: \mathbb{R}^3 \to \mathbb{R}^3$

$\ker P = \{ v \in \mathbb{R}^3 \mid P(v) = 0 \} = \{ (x,y,z) \mid \frac{1}{2}(x+z,\ 2y,\ x+z) = (0,0,0) \}$

$$\underbrace{\left[\begin{array}{ccc|c} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right]}_{\text{matrix of } P!} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x = -s \\ y = 0 \\ z = s \end{array}$$

$= \{ (x,y,z) \mid A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \}$; $A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

$\therefore \ker P = \{ (-s, 0, s) \mid s \in \mathbb{R} \}$

$= \operatorname{span}\{(-1,0,1)\}$ (normal line to h

$\operatorname{im} P = \{ P(v) \mid v \in \mathbb{R}^3 \}$; $\quad P(v) = Av$

$= \{ Av \mid v \in \mathbb{R}^3 \}$

$= \operatorname{col} A$!

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \operatorname{im} P = \operatorname{span}\{ (\frac{1}{2}, 0, \frac{1}{2}),\ (0,1,0) \}$

$\overset{!}{=} W$ (check)

Note: $\dim \ker P + \dim \operatorname{im} P$

$= \dim \ker A + \dim \operatorname{col}(A)$

$= \dim \ker A + \operatorname{rank} A = 3$ (= dimension of domain

$= \dim \mathbb{R}^3$, $\quad P: \mathbb{R}^3 \longrightarrow \ldots$

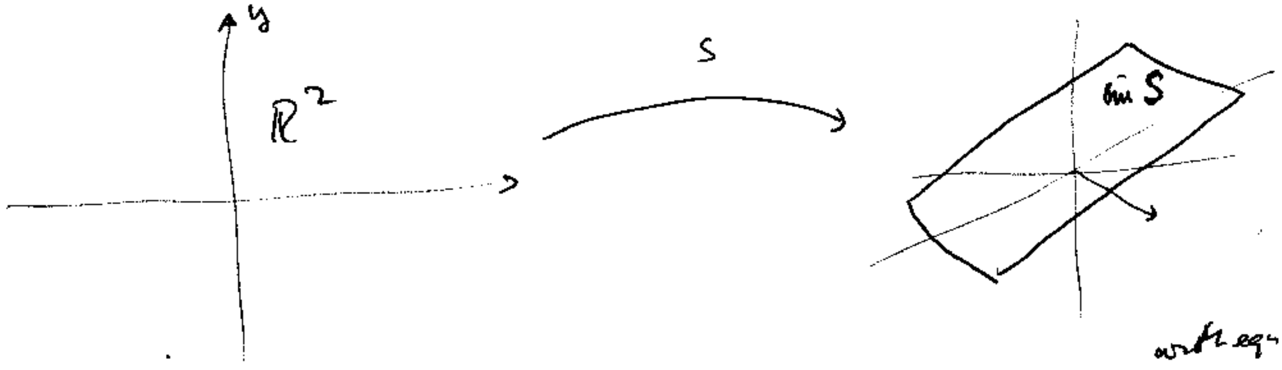
e.g. $S: \mathbb{R}^2 \to \mathbb{R}^3$ def'd by $S(x,y) = (x, y, x+y)$

note: $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $\therefore S(v) = Av$ so S is a l.t.

- $\ker S = \ker A$; $[A|0] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\therefore \ker A = \{(0,0)\}$

- $\operatorname{im} S = \operatorname{col} A = \operatorname{span}\{\underset{v_1}{(1,0,1)}, \underset{v_2}{(0,1,1)}\}$

  is plane through O with normal $v_1 \times v_2$
  $= (1,1,-1)$



S "embedds" $\mathbb{R}^2$ in $\mathbb{R}^3$ as plane with eqn $x + y - z = 0$.

$\dim \ker S + \dim \operatorname{im} S = \dim \ker A + \dim \operatorname{col} A$
$= \#\text{cols } A = 2$

("conservation of dimension").