1. The following is a table of some values of two functions $y = f(x)$ and $y = g(x)$.

x	1	2	3	4
$f(x)$	2	3	1	1
$g(x)$	1	4	3	2

Find

(a) $(f \circ g)(2)$ (1)

(b) $(g \circ f)(1)$ (1)

(c) $(g \circ g)(3)$ (1)

(a) $(f \circ g)(2) = f(g(2)) = f(4) = 1$

(b) $(g \circ f)(1) = g(f(1)) = g(2) = 4$

(c) $(g \circ g)(3) = g(g(3)) = g(3) = 3$

2. Consider the function

$$f(x) = \frac{2\ln(x)}{\ln(x) - 1}$$

(a) What is the domain of the function f? (1)

(b) Find the inverse of the function f. (2)

(c) What is the range of the function f? (1)

(a) $x > 0$ and $x \neq e$

(b) $y = \dfrac{2\ln(x)}{\ln(x) - 1}$

$y(\ln(x) - 1) = 2\ln(x)$

$\ln(x)(y - 2) = y$

$\ln(x) = \dfrac{y}{y-2}$ so $x = e^{\frac{y}{y-2}}$

The inverse function is $y = e^{\frac{x}{x-2}}$

(c) The range of f is the domain of f^{-1}, which is all real numbers except 2.

3. Exponential and logarithmic functions:

(a) What is $\log_3 \frac{1}{\sqrt{3}}$? (1)

(b) Solve for x, if $\log_2(x+1) + \log_2(x-1) = 2$. (2)

(a) Since $\sqrt{3} = 3^{\frac{1}{2}}$, $\frac{1}{\sqrt{3}} = 3^{-\frac{1}{2}}$

So $\log_3 \frac{1}{\sqrt{3}} = \log_3 3^{-\frac{1}{2}} = -\frac{1}{2}$.

(b). $\log_2(x+1) + \log_2(x-1) = 2$

$\log_2[(x+1)\cdot(x-1)] = 2$

$\log_2(x^2-1) = 2$

$x^2-1 = 2^2 = 4$

$x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

But $-\sqrt{5}$ is not in the domain of $\log_2(x+1)$ and $\log_2(x-1)$

So the only solution is $\underline{\underline{x = \sqrt{5}}}$

4. Finding limits:

(a) (1)

$$\lim_{x\to\infty} \frac{2x+5}{x-4}$$

(b) (1)

$$\lim_{x\to 4^+} \ln(x^2-16)$$

(c) (2)

$$\lim_{x\to 0} \frac{2-\sqrt{4+x}}{x}$$

(a) $\lim\limits_{x\to\infty} \frac{2x+5}{x-4} = \lim\limits_{x\to\infty} \frac{2+\frac{5}{x}}{1-\frac{4}{x}}$

as $x \to \infty$, $\frac{5}{x} \to 0$ and $\frac{4}{x} \to 0$

So $\lim\limits_{x\to\infty} \frac{2+\frac{5}{x}}{1-\frac{4}{x}} = \frac{2}{1} = 2$

(b) $\lim\limits_{x\to 4^+} \ln(x^2-16)$

as $x \to 4^+$ $x^2 \to 16^+$ and $x^2-16 \to 0^+$ ~~[illegible]~~

Then $\lim\limits_{x\to 4^+} \ln(x^2-16) = -\infty$ as $\lim\limits_{h\to 0^+} \ln(h) = -\infty$.

(c) $\lim\limits_{x\to 0} \frac{2-\sqrt{4+x}}{x} = \lim\limits_{x\to 0}\left(\frac{2-\sqrt{4+x}}{x} \cdot \frac{2+\sqrt{4+x}}{2+\sqrt{4+x}}\right) =$

$= \lim\limits_{x\to 0} \frac{4-(4+x)}{x(2+\sqrt{4+x})} = \lim\limits_{x\to 0} -\frac{1}{2+\sqrt{4+x}} = -\frac{1}{4}.$

5. Use the definition of the derivative to find the derivative of the function

(a) $y = x^2 + x$ when $x = 1$. (1)

(b) $y = x^3$ when $x = -1$. (1)

(a): $$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 + (1+h) - 2}{h} =$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 + h - 2}{h} = \frac{3h + h^2}{h} =$$

$$= \lim_{h \to 0} (3 + h) = 3$$

(b): $$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \frac{(h-1)^3 - (-1)}{h} =$$

$$= \lim_{h \to 0} \frac{h^3 - 3h^2 + 3h - 1 + 1}{h} = \frac{3h - 3h^2 + h^3}{h} =$$

$$= \lim_{h \to 0} (3 - 3h + h^2) = 3.$$

6. Consider a function $y = f(x)$. Suppose this function satisfies all of the following conditions:

- $f'(x) > 0$ if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$,
- $f'(2) = 0$, $\lim_{x\to\infty} f(x) = 1$, $f(-x) = -f(x)$,
- $f''(x) < 0$ if $0 < x < 3$, $f''(x) > 0$ if $x > 3$.

Sketch the graph of this function. (4)