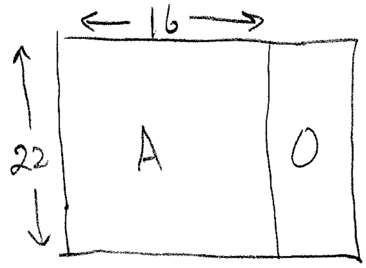
1. If the coefficient matrix A in a homogeneous system of 22 equations in 16 unknowns is known to have rank 5, how many parameters are there in the general solution?



Since rank A = 5, and the system is consistent (it is homogeneous), we know

$$\begin{aligned}\#\text{ parameters} &= \#\text{cols } A - \text{rank } A \\ &= 16 - 5 \\ &= 11\end{aligned}$$

ANSWER $\boxed{11}$

2. For a *nonhomogeneous* system of 2012 equations in 1999 unknowns, answer the following three questions:

I ∘ Can the system be inconsistent?
II ∘ Can the system have infinitely many solutions?
III ∘ Can the system have a unique solution?

Consider the augmented matrix of the system:

$$[A \mid b]$$

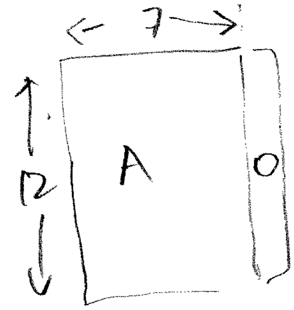
A. Yes, Yes, No.
B. No, No, Yes.
C. Yes, No, Yes.
D. No, Yes, Yes.
E. Yes, Yes, Yes.
F. No, No, No.

(I) Yes - eg $\left[0 \,\middle|\, \begin{matrix}1\\0\\\vdots\end{matrix}\right]$ is inconsistent

(II) Yes, eg $[0 \mid 0]$ has ∞ly many soln (with 1999 parameters)

(III) Yes, if rank A in $[A \mid b]$ is 1999 and it is consistent, the system will have a unique soln. (It is possible for rank $A = 1999$, since rank $A \le \min(1999, 2012) = 1999$.

3. Let A be the 12×7 coefficient matrix of a *homogeneous* linear system, and suppose that this system has the unique solution $0 = (0, \dots, 0) \in \mathbf{R}^7$.



(I) • What is the rank of A?

(II) • Do the columns of A, considered as vectors in $\mathbf{R}^{12}$, span $\mathbf{R}^{12}$?

A. 0, Yes
B. 7, Yes
C. 7, No
D. 5, Yes
E. 5, No
F. 12, Yes

(I) Since the system has a unique sol^n, rank A = # cols A = 7

(II) No: 7 vectors in $\mathbb{R}^2$ (i.e. the cols of A) cannot span $\mathbb{R}^2$, since $\dim \mathbb{R}^{12} = 12 > 7$.

ANSWER $\boxed{C}$

4. Which of the statements below is true for the following system?

$$\begin{array}{rcrcrcr} 2x & - & y & + & 2z & = & 0 \\ x & + & y & - & 2z & = & -2 \\ 3x & - & y & + & z & = & 4 \\ 2x & + & y & - & z & = & 0 \end{array}$$

A. It has no solutions
B. It has an infinite number of solutions
C. It has the trivial solution
D. It has the unique solution (3, 4, 5)
E. It has the solutions $\pm(4,\ 3,\ 1)$
F. It has the unique solution (0, -1, 2)

ANSWER $\boxed{A}$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 1 & 1 & -2 & -2 \\ 3 & -1 & 1 & 4 \\ 2 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & -3 & 6 & 4 \\ 0 & -4 & 5 & 10 \\ 0 & -1 & 3 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & -3 & -8 \\ 0 & 0 & -7 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 8/3 \\ 0 & 0 & 0 & -12\tfrac{2}{3} \end{array}\right]$$

Hence, this system is inconsistent

$$-6 + \frac{7 \times 8}{3} = -6 + 18\tfrac{2}{3}$$

5. The set S of solutions of the following linear system is a subspace of $\mathbf{R}^4$:

$$\begin{array}{rcrcrcrcl} u & - & 2x & + & 3y & + & 4z & = & 0 \\ -u & + & 2x & - & 2y & - & z & = & 0 \end{array}$$

Find a basis for S. We find the set of solutions: (i.e. the gen^l sol^n)

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ -1 & 2 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccccc} & s & & t & \end{array} \\ \left[\begin{array}{cccc|c} 1 & -2 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \end{array}$$

$$\left.\begin{aligned} \therefore (x_1, x_2, x_3, x_4) &= (2s + 5t, s, -3t, t) \\ &= s(2,1,0,0) + t(5,0,-3,1) \end{aligned}\right\} (*)$$

$\therefore S = \operatorname{span}\{\underbrace{(2,1,0,0)}_{v_1}, \underbrace{(5,0,-3,1)}_{v_2}\}$. Hence $\{v_1, v_2\}$ spans S. Moreover, $sv_1 + tv_2 = (0,0,0,0) \iff s = 0$ & $t = 0$ (see $(*)$). Thus $\{(2,1,0,0), (5,0,-3,1)\}$ is also l.i. and hence is a basis of S.

ANSWER $\boxed{\{(2,1,0,0),\ (5,0,-3,1)\}}$

* $\boxed{-2}$ for a serious error that considerably simplifies the question.

6. Suppose $a, c \in \mathbf{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcr} x & & & + & z & = & -1 \\ 2x & + & y & + & z & = & -1 \\ 3x & - & 2y & + & az & = & c \end{array}$$

(You must justify all your answers.)

$2\frac{1}{2}$ a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find rank A and rank$[\,A\,|\,b\,]$ for <u>all</u> values of a and c.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ 3 & -2 & a & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & a-3 & c+3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & a-5 & c+5 \end{array}\right] \quad \frac{1}{2}$$

Hence $\text{rank } A = \begin{cases} 2 & \text{if } a = 5 \quad \frac{1}{2} \\ 3 & \text{if } a \neq 5 \quad \frac{1}{2} \end{cases}$

Consistent with

$$\text{rank}[A|b] = \begin{cases} 2 & \text{if } a = 5 \text{ and } c = -5 \quad \frac{1}{2} \\ 3 & \text{if } a = 5 \text{ and } c \neq -5 \\ 3 & \text{if } a \neq 5. \end{cases} \quad \frac{1}{2}$$

(The last 2 possibilities may be summarized as rank $A = 3$ if $a \neq 5$ OR $c \neq -5$.)

$1\frac{1}{2}$ b) Using part (a), find all values of a and c so that this system has

(i) a unique solution, We need: rank A = rank $[A|b] = 3 = \#$ variables,

so there will be a unique soln iff $a \neq 5$ (and c can be any real number). $\frac{1}{2}$

(ii) infinitely many solutions, or We need rank A = rank $[A|b] < 3$,

so there will be ∞ly many soln iff $a = 5$ and $c = -5$

$\frac{1}{2}$

(iii) no solutions.

We need rank A < rank $[A|b]$, so the system will be inconsistent iff

$a = 5$ and $c \neq -5$. $\frac{1}{2}$

(Answers in (b) may simply be consistent with incorrect answers in (a).)

2 **6c).** In case b(ii) above, give a complete geometric description of the set of solutions.

If $a=5$, $c=-5$, $[A \mid b] \sim \begin{array}{c} \begin{array}{ccc} & & s \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$; $x=-1-s$, $y=1+s$, $z=s$, $s \in \mathbb{R}$.

consistent with

and so the gen'l solution is

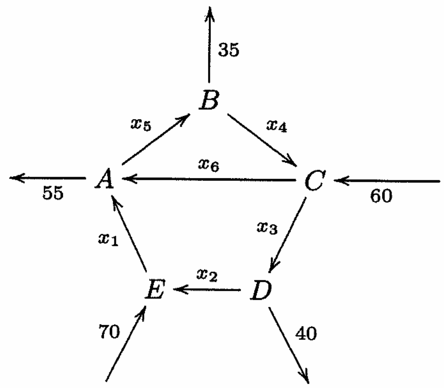
$\{(-1-s, 1+s, s) \mid s \in \mathbb{R}\}$

$= \{(-1,1,0) + s(-1,1,1) \mid s \in \mathbb{R}\}$. $(\frac{1}{2})$

This is the line $(\frac{1}{2})$ through $(-1,1,0)$ $(\frac{1}{2})$ with direction vector $(\frac{1}{2})$ $(-1,1,1)$.

consistent with

7. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(You must justify all your answers.)

3½ a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 6$.

N.B. (Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

We consider the equation # cars in = # cars out (/min) for each intersection:

	# cars in	=	# cars out
A:	$x_1 + x_6$	=	$55 + x_5$
B:	x_5	=	$35 + x_4$
C:	$x_4 + 60$	=	$x_3 + x_6$
D:	x_3	=	$x_2 + 40$
E:	$x_2 + 70$	=	x_1

5@½ =2½

Moreover, since the streets are one-way and x_i denotes a (whole) number of cars, ½ $x_i \geq 0$, $i = 1, \ldots, 6$ &

½ $x_i \in \mathbb{Z}$, "

$1\frac{1}{2}$ **7(b).** The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\begin{array}{c} \begin{array}{cccccc} & & & & s & t \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 1 & 55 \\ 0 & 1 & 0 & 0 & -1 & 1 & -15 \\ 0 & 0 & 1 & 0 & -1 & 1 & 25 \\ 0 & 0 & 0 & 1 & -1 & 0 & -35 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$\begin{aligned} x_1 &= 55 + s - t \\ x_2 &= -15 + s - t \\ x_3 &= 25 + s - t \\ x_4 &= -35 + s \\ x_5 &= s \\ x_6 &= t \end{aligned} \qquad ; \ s, t \in \mathbb{R}$$

6 @ 1/4 (allow <u>one</u> error) $= 1\frac{1}{2}$

1 **7(c).** If $\overline{ED}$ were closed due to roadwork, find the minimum flow along $\overline{AC}$, <u>**using your results from (b).**</u>

Now $\overline{ED}$ is closed $\Leftrightarrow$ $x_2 = 0$, so $s - t = 15$.

We implement the constraints $(x_i \geq 0)$: $x_1 = 70$ ✓

$x_2 = 0$ ✓ $\quad x_3 = 40$ ✓ $\quad x_4 \geq 0 \Leftrightarrow \underline{s \geq 35}$

$x_5 \geq 0 \Leftrightarrow s \geq 0$ ✓ $\quad x_6 \geq 0 \Leftrightarrow t \geq 0$. But

$t = s - 15$ so we need $\underline{s \geq 15}$

But $s \geq 35$ is the strongest constraint, so indeed

$t = s - 15 \geq 35 - 15 \geq 20$. Hence

the flow along ED, namely $x_6 = t$, is at least <u>20</u> cars/min. $\frac{1}{2}$

$\frac{1}{2}$ – justification.

8. [Bonus] If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any 2×2 matrix, and the vectors $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are linearly independent, prove carefully that $\operatorname{rank} A = 2$.

(You cannot choose the matrix A - your proof must work for ***every*** *2×2 matrix with the property above, i.e. every 2×2 matrix with "independent columns".)*

Since $\left\{\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right\}$ is independent, $\begin{bmatrix} a \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We treat 2 cases: (I) $a \neq 0$ & (II) $a = 0$ and $c \neq 0$.

(I) $a \neq 0$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} 1 & b/a \\ 0 & d - \frac{bc}{a} \end{bmatrix}$. If $d = \frac{bc}{a}$, then $\begin{bmatrix} b \\ d \end{bmatrix} = \frac{b}{a}\begin{bmatrix} a \\ c \end{bmatrix}$, and $\left\{\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right\}$ is l.d., a contradiction. Hence $d - \frac{bc}{a} \neq 0$ and so

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \cdots \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$, so $\operatorname{rank}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2$.

(II) $a = 0$ and $c \neq 0$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \sim \begin{bmatrix} 1 & d/c \\ 0 & b \end{bmatrix}$$

If $b = 0$, then $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix} = \frac{d}{c}\begin{bmatrix} 0 \\ c \end{bmatrix} = \frac{d}{c}\begin{bmatrix} a \\ c \end{bmatrix}$, which is a contradiction. Hence $b \neq 0$ and

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}$ and $\operatorname{rank}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2$.

(1) - some correct idea + (1) some correct progress.
+ (1) all cases considered + (1) clearly presented.