1. Which of the following subsets of $\mathbf{R}^4$ are closed under (the standard operation of) multiplication by scalars?

A. $\{(a,\ b,\ c,\ d)|\ a\,b\,c = 0\}$
B. $\{(a,\ b,\ c,\ d)|\ a = 1, b = 0\ \text{ and } c + d = 0\}$
C. $\{(a,\ b,\ c,\ d)|\ a > 1 \text{ and } b < 1\}$
D. $\{(a,\ b,\ c,\ d)|\ a > 0 \text{ and } b > 0\}$
E. $\{(a,\ b,\ c,\ d)|\ a - b + 2c = 0\}$

Recall for $s \in \mathbb{R}$ we have $s(a,b,c,d) = (sa, sb, sc, sd)$

(A) If $abc = 0$ then $(sa)(sb)(sc) = s^3 abc = 0$, so (A) is closed

(B) If $a = 1$, then $sa \neq 1$ in general, e.g. for $s = 2$, so (B) is not closed

(C) If $a > 1$, then $sa > 1$ in general, e.g. for $s = -1$, so (C) is not closed.

(D) Not closed, same argument as in (C)

(E) $(sa) - (sb) + 2(sc) = s(a - b + 2c) = 0$, so (E) is closed.

ANSWER $\boxed{A, E}$

2. Which of the following are subspaces of $\mathbf{F}[\mathbf{R}] = \{f \mid f\colon \mathbf{R} \to \mathbf{R}\}$?

$U = \{f \in \mathbf{F}[\mathbf{R}] \mid f(-1)f(1) = 0\}$
$V = \{f \in \mathbf{F}[\mathbf{R}] \mid f(1) + f(2) = 0\}$
$S = \{f \in \mathbf{F}[\mathbf{R}] \mid f(x) = f(-x), \forall x \in \mathbf{R}\}$
$T = \{f \in \mathbf{F}[\mathbf{R}] \mid f(1) \leq 0\}$

$\boxed{U}$: not closed under addition: Let $f \in F[\mathbb{R}]$ be given by $f(x) = x + 1$. Then $f(-1) = 0$, so $f \in U$. Also, consider the function $g \in F[\mathbb{R}]$, $g(x) = x - 1$. Then $g \in U$ since $g(1) = 0$ so $g(1)g(-1) = 0$. But $(f+g)(-1) = f(-1) + g(-1) = 0 - 2 = -2$, $(f+g)(1) = f(1) + 0 = 2 + 0 = 2$, so $(f+g)(1)(f+g)(-1) = (-2)(2) = -4 \neq 0$, and $f + g \notin U$.

$\boxed{V}$ is a subspace (apply subspace test)

$\boxed{S}$ yes, is a subspace (apply subspace test)

$\boxed{T}$ no, not closed under scalar multiplication:
Let $f \in F[\mathbb{R}]$, $f(x) = -1$ (constant function!), so $f \in T$. But $-2f$ is the constant function $(-2)(-1) = 2$, so $-2f \notin T$.

ANSWER $\boxed{V, S}$

two

three

3. Which ~~two~~ of the following statements are true?

I. The span of any vector in $\mathbf{R}^3$ is a line through the origin.

II. The span of any two distinct vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$.

III. A set of vectors $\{u, v, w\}$ in a vector space spans V if every vector on V is a linear combination of $u, u+v$ and $u+v+w$.

~~IV. Any spanning set for $\mathbf{M}_{2,2}$ contains at least four elements.~~

V. The set $\{(1,1),(2,3)\}$ spans $\mathbf{R}^2$.

(I) False: The span of $0_{\mathbb{R}^3}$ is $\{0_{\mathbb{R}^3}\}$.

(II) False: The span of $0_{\mathbb{R}^2}$ and a $0 \neq d \in \mathbb{R}^2$ is the line through d.

(III) True since every $x \in V$ has the form $x = au + b(u+v) + c(u+v+w)$
$= (a+b+c)u + (b+c)v + cw$ for $a, b, c \in \mathbb{R}$.

(IV) True by the Fundamental Theorem

(V) True:
$(a,b) = (3a-2b)(1,1) + (b-a)(1,1)$

ANSWER $\boxed{\text{III}, \cancel{\text{IV}}\ \text{V}}$

4. For which value of s does the vector (6, 3, s) belong to the subspace of $\mathbf{R}^3$ spanned by $(1,2,3)$ and $(0,1,2)$?

We need to find $x, y \in \mathbb{R}$ such that
$x(1,2,3) + y(0,1,2) = (6,3,s)$.

Equivalently, for which s is
$$\left\{\begin{aligned} x \quad &= 6 \\ 2x + y &= 3 \\ 3x + 2y &= s \end{aligned}\right\}$$
solvable.

We get $x = 6$, $y = 3 - 2x = 3 - 12 = -9$, so $s = 3x + 2y = 3 \cdot 6 + 2 \cdot (-9) =$
$= 18 - 18 = 0$

ANSWER $\boxed{s = 0}$

5. If we give $X = \mathbf{R}^2$ the *non-standard* operations

$$(x,\ y) \oplus (x',\ y') = (x + x' - 1,\ y + y' + 2) \quad \text{(vector addition)}$$

and

$$k \odot (x,\ y) = (kx - k + 1,\ ky + 2k - 2) \quad \text{(multiplication by scalars),}$$

then X *is* a real vector space.

- What is the zero vector of X?
- If $\mathbf{v} = (x,\ y)$ is in X then what is $-\mathbf{v}$?

ANSWER $\boxed{0 = (1, -2) \qquad -\mathbf{v} = (-x+2, -y-4)}$

Let $0 = (x', y')$. Then $(x, y) \oplus (x+x'-1, y+y'+2) = (x, y)$

so $\left\{\begin{array}{l} x + x' - 1 = x \\ y + y' + 2 = y \end{array}\right\} \iff \begin{array}{l} x' = 1 \\ y' = -2 \end{array}$

$-v = (-1)\odot(v)$, in general, so, if $v = (x, y)$ we get

$-v = (-1)\odot = ((-1)x - (-1) + 1,\ (-1)y + 2(-1) - 2)$

$= (-x + 1 + 1,\ -y - 2 - 2) = (-x + 2, -y - 4)$

6. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x - y + z = 0\}$

2 a) Is W a subspace of $\mathbf{R}^3$?

$2\frac{1}{2}$ b) Find a spanning set for W.

$1\frac{1}{2}$ c) Give a complete geometric description of W.
(You must justify your answers.)

$1\frac{1}{2}$ - justification

a) Yes $\frac{1}{2}$ W is a subspace of $\mathbb{R}^3$ because it is a plane through the origin (see (c)). OR Yes, because it is the span of 2 vectors in $\mathbb{R}^3$ (see (b)). (OR, run the Subspace test!)

b) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x - y + z = 0\}$

$= \{(y-z, y, z) \mid y, z \in \mathbb{R}\}$

$= \text{span}\{(1,1,0), (-1,0,1)\}$.

Hence $\underline{\{(1,1,0), (-1,0,1)\}}$ is a spanning set for W.

1 - any correct answer

1 - Justification.

c) W is $\underset{\frac{1}{2}}{\underline{\text{a plane}}}$ $\underset{\frac{1}{2}}{\underline{\text{through } 0}}$ with $\underset{\frac{1}{2}}{\underline{\text{normal } (1,-1,1)}}$.

7. Consider the vector space $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$, with the standard operations. Recall that the zero of $\mathbf{F}(\mathbf{R})$ is the function that has the value 0 for all $x \in \mathbf{R}$.

Let $U = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) = f(-1)\}$ be the subspace of functions which have the same value at $x = -1$ and $x = 1$.

Define functions g, h, j and $k \in \mathbf{F}[\mathbf{R}]$ by

$$g(x) = 2x^3 - x^2 - 2x + 1, \quad h(x) = x^3 + x^2 - x + 1,$$
$$k(x) = -x^3 + 5x^2 + x + 1 \quad \text{and} \quad j(x) = x^3 - x, \qquad \forall x \in \mathbf{R}.$$

1 a) Show that g and h belong to U.

1½ b) Show that $k \in \operatorname{span}\{g, h\}$.

2 c) Show that $j \notin \operatorname{span}\{g, h\}$.

1½ d) Show that $\operatorname{span}\{g, h\} \neq \operatorname{span}\{g, h, j\}$.

(You must justify your answers.)

a) Since $g(1) = 2-1-2+1 = 0$ and $g(-1) = -2-1+2+1 = 0$, $g(1) = g(0)$, so $g \in U$.
Similarly, $h(1) = 2 = h(-1)$, so $h \in U$. 1

b) By equating coefficients of $1, x, x^2$, and x^3 in the expression
½ $k = a g + b h$, we find $a = -2$, ½ and $b = 3$. Then we
check: $-2k(x) + 3h(x) = -2(2x^3 - x^2 - 2x + 1) + 3(x^3 + x^2 - x + 1)$
$= x^3 + 5x^2 + x + 1 = k(x), \quad \forall x \in \mathbb{R}$.
½. $\therefore k \in \operatorname{span}\{g, h\}$.

c) Suppose $j = ag + bh$. 1 Then $x^3 - x = (2a+b)x^3 + (b-a)x^2 - (2a+b)x + a + b$, $\forall x \in \mathbb{R}$. For $x = 0$, this yields $a + b = 0$; for $x = 1$, we obtain $0 = 2b$
So $a = b = 0$. But then $j(x) = 0$, contradicting $j(2) = 6$. Hence
$j \notin \operatorname{span}\{g, h\}$ + 1 any correct justification

d) Since $j = 0g + 0h + 1j \in \operatorname{span}\{g, h, j\}$, but $j \notin \operatorname{span}\{g, h\}$, $\operatorname{span}\{g, h\} \neq \operatorname{span}\{g, h, j\}$
1 – knowing how to show 2 sets are not equal ½ – doing it.

8. [Bonus] Let $\mathbf{E} = \{\text{“}ax+by+cz = d\text{”} \mid a, b, c, d \in \mathbf{R}\}$ be the set of linear equations with real coefficients in the variables x, y and z. Equip $\mathbf{E}$ with the usual operations on equations that you learned in high school: addition of equations, denoted here by "$\oplus$" and multiplication by scalars, denoted here by "$\circledast$", as follows:

$$\text{“}ax+by+cz=d\text{”} \oplus \text{“}ex+fy+gz=h\text{”} = \text{“}(a+e)x+(b+f)y+(c+g)z=d+h\text{”}$$

and

$$\forall k \in \mathbf{R}, \quad k \circledast \text{“}ax+by+cz=d\text{”} = \text{“}ka\,x+kb\,y+kc\,z=k\,d\text{”}.$$

You may assume without proof that $\mathbf{E}$ is a vector space.

Find a spanning set for $\mathbf{E}$.

(You must justify your answer.)

Since "$ax+by+cz=d$" $= a \circledast (\text{“}x=0\text{”}) \oplus b \circledast (\text{“}y=0\text{”})$
$\oplus\, c \circledast (\text{“}z=0\text{”}) \oplus d \circledast (\text{“}0=1\text{”})$,

we see that $E = \{\text{“}ax+by+cz=d\text{”} \mid a, b, c, d \in \mathbb{R}\}$
$= \text{span}\{\text{“}x=0\text{”}, \text{“}y=0\text{”}, \text{“}z=0\text{”}, \text{“}0=1\text{”}\}$

Thus $\underline{\{\text{“}x=0\text{”}, \text{“}y=0\text{”}, \text{“}z=0\text{”}, \text{“}0=1\text{”}\}}$ is a spanning set for E.

$1\frac{1}{2}$ - justification.

$1\frac{1}{2}$ - correct, well written answer