

1. If $\{u, v, w\}$ is a set of vectors in a vector space V , and a, b , and c are scalars, which of the following statements are true?

- I. If $a = b = c = 0$ implies $au + bv + cw = 0$, then $\{u, v, w\}$ is linearly independent.
- II. If $au + bv + cw = 0$ implies $a = b = c = 0$, then $\{u, v, w\}$ is linearly independent.
- III. If none of the vectors u, v or w is a multiple of any other vector in $\{u, v, w\}$, then $\{u, v, w\}$ is linearly independent.
- IV. If $au + bv + cw = 0$ can occur only when $a = b = c = 0$, then $\{u, v, w\}$ is linearly independent.

- A. I & II
- B. I & III
- C. II & III
- D. I & IV
- E. II & IV**
- F. III & IV

I is false: the statement " $a=b=c=0 \Rightarrow au+bv+cw=0$ " holds for any vectors u, v, w (e.g. $u=v=w=0$).

II is true: it is the definition of linear independence

III is false: $\{(1,0), (0,1), (1,1)\}$ satisfies the given condition, but is l. dependent.

IV is true: this is simply a restatement of the definition: "A implies B" means A can only occur when B does too.

2. For what value of a is the set of vectors $\{(1, 1, 1), (1, 0, 2), (1, a, 1)\}$ linearly dependent?

- A. -1
- B. 2
- C. 0
- D. 1**
- E. -1/2
- F. -2

We seek values of a st. there are scalars b, c

s.t. $b(1,1,1) + c(1,0,2) = (1,a,1)$

i.e. $b + c = 1$ (1)

$b = a$ (2)

$b + 2c = 1$ (3)

(1) & (3) $\Rightarrow c=0$, and $b=1$, and (2) then implies $a=b=1$.

3. In a vector space V , suppose $\{u, v\}$ is linearly independent and w is such that $\{u, v, w\}$ is linearly dependent. Which of the following is **ALWAYS** true?

- A. $\{u, w\}$ is linearly dependent
- B. $\{v, w\}$ is linearly dependent
- C. $u \in \text{span}\{v, w\}$
- D. $v \in \text{span}\{u, w\}$
- E. $w \in \text{span}\{u, v\}$
- F. $\{u, u+v, w\}$ is linearly independent

A could be false. e.g. $u=(1,0)$, $v=(0,1)$ and $w=(1,1)$
 $\{u, v\}$ is l.i. and $\{u, w\}$ is l.i. too

B could be false: Same example as for A.

C " " " " " " " " "

D " " " " " " " " "

E is true! It's a theorem we proved in class:

if $\{u, v\}$ is l.i. then $\{u, v, w\}$ is l.i. \Leftrightarrow

$w \notin \text{span}\{u, v\}$.

F. could be false. See the example for A

4. Let $u = (1, 0, -1)$, and $U = \{w \in \mathbb{R}^3 \mid \text{proj}_u w = 0\}$.

a) Show that $U = \{(x, y, z) \in \mathbb{R}^3 \mid x - z = 0\}$; (1)

2 1/2 b) Give a complete geometric description of U .
Is U a subspace of \mathbb{R}^3 ? (1/2)

c) Show that $\{(1, 0, 1), (0, 1, 0)\}$ spans U . (1)

1/2 d) What is the dimension of U ? (1/2)

a) Note that (for $w = (x, y, z)$) $\text{proj}_u w = 0$ is the equation $\frac{u \cdot w}{\|u\|^2} u = 0$. But as $u \neq 0$, $\frac{u \cdot w}{\|u\|^2} u = 0$

$$\Leftrightarrow \frac{u \cdot w}{\|u\|^2} = 0 \Leftrightarrow u \cdot w = 0 \Leftrightarrow (1, 0, -1) \cdot (x, y, z) = 0$$

$$\Leftrightarrow x - z = 0. \text{ Hence } U = \{(x, y, z) \in \mathbb{R}^3 \mid x - z = 0\}.$$

b) U is the plane through the origin with normal vector $(1, 0, -1)$. $3 \times (1/2)$

U is a subspace of \mathbb{R}^3 because it is a plane through the origin. (or: see (c) $U = \text{span}\{(1, 0, 1), (0, 1, 0)\}$ and so by a theorem in class is a subspace of \mathbb{R}^3). (1/2) - reason

c) From (a) $U = \{(z, y, z) \mid y, z \in \mathbb{R}\} = \{z(1, 0, 1) + y(0, 1, 0) \mid y, z \in \mathbb{R}\} = \text{span}\{(1, 0, 1), (0, 1, 0)\}$. And $\{(1, 0, 1), (0, 1, 0)\}$ spans U . (1)

d) By (c), $\{(1, 0, 1), (0, 1, 0)\}$ spans U . Moreover, $(1, 0, 1)$ & $(0, 1, 0)$ are l.o.i. because neither (of these 2 vectors) is a multiple of the other. Hence $\{(1, 0, 1), (0, 1, 0)\}$ is a basis for U .

So $\dim U = 2$.

(1/2) + (1) - justification

5. Let $F[0, \pi] = \{f \mid f : [0, \pi] \rightarrow \mathbf{R}\}$ be the vector space of real-valued functions defined on $[0, \pi]$. Define three functions in $F([0, \pi])$ by

$$f(x) = 1, \quad g(x) = \sin 2x, \quad \text{and} \quad h(x) = \cos x, \quad \forall x \in [0, \pi],$$

and let $W = \text{span}\{f, g, h\}$.

2a) Show that $\{f, g, h\}$ is linearly independent.

1b) Use (a) to show that $\{f+g, f+h\}$ is linearly independent.

1/2 c) Give a spanning set S for W with $S \neq \{f, g, h\}$.

(knowing what to do)

1/2 d) What is the dimension of W ?

a) Suppose scalars a, b, c satisfy $af + bg + ch = 0$.

Then $a + b \sin 2x + c \cos x = 0, \quad \forall x \in [0, \pi]$. (1)

So for $x=0$, we obtain $a + c = 0$

$x = \frac{\pi}{4}$ " $a + b = 0$

$x = \frac{\pi}{2}$ " $a = 0$

These clearly imply $a=b=c=0$.

Hence $\{f, g, h\}$ is l.i. (1) any set of eqns implying (consistent with $af+bg+ch=0$)

b) Suppose $a(f+g) + b(f+h) = 0$. (1/2) Then $(a+b)f + bg + bh = 0$.

But $\{f, g, h\}$ is l.i., so $a+b=0$, $b=0$, and $b=0$. This implies $a=b=0$, so $\{f+g, f+h\}$ is l.i. (1/2)

c) Since none of f, g, h is the zero function, $S = \{f, g, h, 0\} \neq \{f, g, h\}$.
(1) - any correct answer + (1/2) justification.
 But $\text{span}\{f, g, h, 0\} = \text{span}\{f, g, h\} = W$, since $0 \in W$ already.
 (or e.g. $S = \{f, g, h, f+g\}$ spans W but is not the same set as $\{f, g, h\}$.)

d) By definition of W , $\{f, g, h\}$ spans W , and by (a) is l.i. Hence $\{f, g, h\}$ is a basis of W and so $\dim W = 3$.

(1/2)

(1) - justification

6.

- a) Give an example of 3 vectors u, v, w in \mathbf{R}^2 , such that $\{u, v, w\}$ spans \mathbf{R}^2 , and $\{u, v\}$ also spans \mathbf{R}^2 .

$$\text{Let } u = (1, 0), v = (0, 1) \text{ and } w = (1, 1).$$

$$\text{Then } \text{span}\{u, v\} = \mathbb{R}^2 \quad \left(\begin{array}{l} \forall (x, y) \in \mathbb{R}^2, \\ (x, y) = xu + yv \end{array} \right)$$

$$\text{and } \text{span}\{u, v, w\} = \mathbb{R}^2 \quad \left(\begin{array}{l} \forall (x, y) \in \mathbb{R}^2, \\ (x, y) = xu + yv + 0w. \end{array} \right)$$

$$\textcircled{1} - \text{span}\{u, v, w\} = \mathbb{R}^2$$

$$\textcircled{2} - \text{span}\{u, v\} = \mathbb{R}^2$$

+ $\textcircled{1}$ - correct justification

- b) If U is vector space with u, v, w in U , and $\{u, v, w\}$ is linearly independent, show that $\{u, v\}$ must also be linearly independent. (Do not give an example: use the definition of independence to show that the statement is true in every vector space U .)

Suppose a, b are scalars st $au + bv = 0$. $\textcircled{1}$

Then $au + bv + 0w = 0$. But $\{u, v, w\}$ is

l.i., so $a=0, b=0$ and $0=0$. This implies

$a=b=0$, so $\{u, v\}$ is l.i.

$\textcircled{1}$

7. [Bonus]

- a) Suppose that u, v are two non-zero vectors in \mathbf{R}^4 such that $u \cdot v = 0$. Prove carefully that $\{u, v\}$ is linearly independent.

Suppose a, b are scalars with $au + bv = 0$. (*)

Then $u \cdot (au + bv) = u \cdot 0 = 0$, i.e.

$$\|u\|^2 a + b(u \cdot v) = 0. \quad \text{But } u \cdot v = 0, \text{ so}$$

$$\|u\|^2 a = 0. \quad \text{Since } u \neq 0, \|u\|^2 \neq 0, \text{ so } a = 0.$$

Then we obtain from (*) that $bv = 0$. But

since $v \neq 0$, b must be 0. Hence $a = b = 0$, so

$\{u, v\}$ is l.o.i. (1) - correct and well-written

- b) Give an example of two non-zero vectors u and v in \mathbf{R}^4 such that $\{u, v\}$ is linearly independent but $u \cdot v \neq 0$.

let $u = \frac{1}{2}(1, 0, 0, 0)$ and $v = \frac{1}{2}(1, 1, 0, 0)$.

Then neither u nor v is a multiple of the other,

so $\{u, v\}$ is l.o.i., but $u \cdot v = 1 \neq 0$.

(1/2) - correct example

(1/2) - justification, well-written.