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1. If C is a $n \times 4$ matrix and $D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and then the second column of the matrix CD is $= C_1$ (col 2 of D)

- A. the same as the second column of C .
- B. the sum of the first and second columns of C .
- C. the sum of the second and fourth columns of C .
- D. the same as the first column of D .
- E. the same as the third row of D .
- F. the sum of the first and the third columns of C .

$$CD = [C_1 C_2 C_3 C_4] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \downarrow & \downarrow \\ & C_2 \\ \downarrow & \downarrow \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 1 & 2 \end{bmatrix}$. Then the second row of A^{-1} is:

- A. $[-4 \ 1 \ 1]$
- B. $[2 \ 1 \ -1]$
- C. $[3 \ -1 \ 0]$
- D. $[-1 \ 0 \ 2]$
- E. $[2 \ 0 \ -1]$
- F. $[0 \ -1 \ 1]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{array} \right]$$

(we could stop here since the second row won't change)

2'' Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$. Then the second row of A^{-1} is:

- A. $[0 \ -1 \ 1]$
- B. $[0 \ -1 \ 0]$
- C. $[2 \ -1 \ 0]$
- D. $[-1 \ 1 \ 1]$
- E. $[2 \ 0 \ -1]$
- F. $[0 \ -1 \ 1]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right]$$

Again, we could stop here as subsequent row ops won't change the second row..)

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array} \right]$$

3. Compute $\left[\begin{array}{cc|cc} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]^{2012} = A$

A. $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & -6036 & 0 \\ 0 & 1 & 0 & 6036 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 & 0 & -3^{2012} \\ 0 & 1 & 3^{2012} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 2K \\ 0 & I \end{bmatrix}$$

$$A^3 = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 2K \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 3K \\ 0 & I \end{bmatrix}; \text{ so}$$

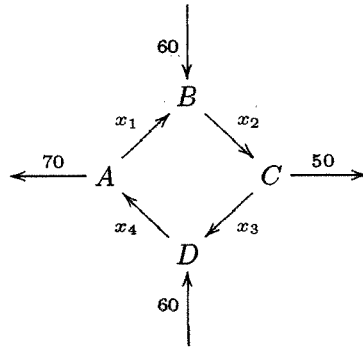
(B.) $\begin{bmatrix} 1 & 0 & 0 & -6036 \\ 0 & 1 & 6036 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$A^n = \begin{bmatrix} I & nK \\ 0 & I \end{bmatrix}$$

D. $\begin{bmatrix} 1 & 0 & 0 & 6036 \\ 0 & 1 & -6036 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 1 & 0 & 0 & 3^{2012} \\ 0 & 1 & -3^{2012} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the **one way streets**, and the numbers refer to the number of cars observed to enter A or leave B, C and D during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



- (2) a) Write down a system of linear equations which describes the traffic flow, together with all the constraints on the variables x_i , $i = 1, \dots, 4$. (Do not simply copy out the equations implicit in (b). You will not get any marks if you do this. Do not perform any operations on your equations: this is done for you in (b)!)

Intersection		Flow in	=	Flow out	}
A	x_4	=	$70 + x_1$	4 @ $\frac{1}{4}$	
B	$x_1 + 60$	=	x_2		
C	x_2	=	$x_3 + 50$		
D	$x_3 + 60$	=	x_4		

($\frac{1}{2}$) Need $x_i \geq 0$ (one way streets)

($\frac{1}{2}$) $x_i \in \mathbb{Z}$ (whole numbers of cars)

$$i = 1, \dots, 4$$

(Actually, the correct RRE form, and correct answers are below in brackets. You will not lose marks if you assumed the RRE form as given on the left!

(Question 4 continued)

Suppose
The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -70 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & 1 & -1 & -40 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left(\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -70 \\ 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \right)$$

Give the general solution. (Ignore the constraints at this point.)

$$x_1 = -70 + \lambda$$

$$x_2 = 10 + \lambda$$

$$x_3 = -40 + \lambda$$

$$x_4 = \lambda$$

; $\lambda \in \mathbb{R}$

$$4 \times \frac{1}{4} = 1$$

$$x_1 = -70 + \lambda$$

$$x_2 = -10 + \lambda$$

$$x_3 = -60 + \lambda$$

$$x_4 = \lambda$$

; $\lambda \in \mathbb{R}$

Using (b) and the constraints from (a), find the minimum traffic flows along

(i) \overline{BC} , and

(ii) \overline{CD} .

$$x_1 \geq 0 \Leftrightarrow \lambda \geq 70$$

$$x_3 \geq 0 \Leftrightarrow \lambda \geq 40$$

$$\therefore \lambda \geq 70$$

$$x_2 \geq 0 \Leftrightarrow \lambda \geq -10$$

$$x_4 \geq 0 \Leftrightarrow \lambda \geq 0$$

(i) The flow along \overline{BC} is $x_2 = 10 + \lambda \geq \underline{80}$

(ii) The flow along \overline{CD} is $x_3 = -40 + \lambda \geq \underline{30}$

(i) Correct answer $\frac{1}{2}$ + Justification 1

(ii) ditto.

$$\left. \begin{array}{l} x_1 \geq 0 \Leftrightarrow \lambda \geq 70 \\ x_2 \geq 0 \Leftrightarrow \lambda \geq 10 \\ x_3 \geq 0 \Leftrightarrow \lambda \geq 60 \\ x_4 \geq 0 \Leftrightarrow \lambda \geq 0 \end{array} \right\} \therefore \lambda \geq 70$$

Hence $x_2 = -10 + \lambda \geq 60$

$x_3 = -60 + \lambda \geq 10$

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5. Let $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 a) Find the reduced row echelon form of A .

2 b) Find a basis for $\ker A$.

2 c) Is A invertible?

2 d) [Bonus: 2pts] Extend your basis of $\ker A$ to a basis of \mathbb{R}^4 , if necessary.

2^a) $\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

① correct

① - some justification

2 b) $(\tilde{A} | 0) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$

$$x_1 = -\Delta$$

$$x_2 = -\Delta$$

$$x_3 = 0$$

$$x_4 = \Delta$$

① correct or consistent with (a)
① justification (work shown)

e. $\ker A = \{ \Delta(-1, -1, 0, 1) \mid \Delta \in \mathbb{R} \}$

$= \text{span} \{ (-1, -1, 0, 1) \}$. Moreover, $(-1, -1, 0, 1)$

is non-zero $\therefore \{ (-1, -1, 0, 1) \}$ is a basis for $\ker A$.

(Students can simply present the "basic solution" $(-1, -1, 0, 1)$ and stop there, since a theorem from class asserts these give a basis of $\ker A$.)

2 c) **No**, since $\text{rank } A = 3 < 4$ (A is 4×4 ! Justification ①)

d) Note that $\text{rank} \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4$, so

② - any correct extension

①^{1/2} - Justification

$\{ (-1, -1, 0, 1), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) \}$ is a basis \mathbb{R}^4

6. Suppose A is an $n \times n$ matrix and that,

there is a non-zero vector $x \in \mathbb{R}^n$, for which $Ax = 0$.

State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example (with numbers!).
- If you say the statement is true, you must give a clear explanation, for example by quoting a theorem presented in class.

a) The rank of A is n .

$$\text{eg } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boxed{\text{False}}$$

but rank $A = 1 < 2$ (A is 2×2 here)

b) The columns of A are linearly dependent.

True, since $Ax = 0$ has a unique $\boxed{\text{TRUE}}$

(zero) solution \Leftrightarrow cols. of A are l.o.s. (from class)

c) There is $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent.

We know ^{from class that} $Ax = b$ is consistent $\boxed{\text{TRUE}}$

$\forall b \in \mathbb{R}^n \Leftrightarrow A$ is invertible

$$\Leftrightarrow Ax = 0 \Rightarrow x = 0$$

Hence $\exists b$ st $Ax = b$ is inconsistent

Each part ① - correct answer + some justification.
 ① Justification (Explanation or proper counterexample)
 Correct

7. Let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and define a subspace $C \subseteq M_{2,2}$ by

$$C = \{A \in M_{2,2} \mid AJ = JA\}.$$

(I) a) Show that

$$C = \left\{ \begin{bmatrix} a & -c \\ c & a \end{bmatrix} \mid a, c \in \mathbb{R} \right\}.$$

(Hint: Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute AJ and JA , and find the general solution to system of the equations in a, b, c and d that guarantees $AJ = JA$.)

(2) b) Find a basis for C , and hence find $\dim C$.

$$(1) a) AJ = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = JA = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix} \Leftrightarrow$$

$$b = -c, \quad -a = -d, \quad d = a \quad \& \quad b = -c$$

$$\therefore b = -c, \quad a = d. \quad \therefore A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}, \text{ for } a, c \in \mathbb{R}$$

$$\text{So } C = \left\{ \begin{bmatrix} a & -c \\ c & a \end{bmatrix} \mid a, c \in \mathbb{R} \right\} \quad (1)$$

$$b) C = \text{span} \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_I, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_J \right\}, \text{ and}$$

$$aI + cJ = 0 \Leftrightarrow \begin{bmatrix} a & -c \\ c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow a = c = 0 \quad \therefore \{I, J\} \text{ is}$$

l.i.c. and spans C , so is a basis of C .

Hence $\dim C = 2$.

(1) any correct basis (2) Justification (2) Dimension (consistent w/ basis)

(Same scheme as other question 7)

7. Suppose $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and we define a subspace $\mathbf{T} \subseteq \mathbf{M}_{22}$ by

$$\mathbf{T} = \{A \in \mathbf{M}_{22} \mid AK = KA\}.$$

a) Show that $\mathbf{T} = \left\{ \begin{bmatrix} a & c \\ c & a \end{bmatrix} \mid a, c \in \mathbf{R} \right\}$.

(Hint: Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute AK and KA , and find the general solution to system of the equations in a, b, c and d that guarantees $AK = KA$.)

$$a) AK = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} = KA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\Leftrightarrow b=c, a=d, (d=a, c=b).$$

$$\therefore A = \begin{bmatrix} a & c \\ c & a \end{bmatrix}, a, c \in \mathbf{R}, \text{ so}$$

$$\mathbf{T} = \left\{ \begin{bmatrix} a & c \\ c & a \end{bmatrix} \mid a, c \in \mathbf{R} \right\}.$$

b) Find a basis for \mathbf{T} , and hence find $\dim \mathbf{T}$. From (a), $\mathbf{T} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$
 $= \text{span} \{I, K\}$. Moreover, $aI + cK = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \begin{bmatrix} a & c \\ c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Leftrightarrow a=c=0$. Hence $\{I, K\}$ is linearly independent.

$\{I, K\}$ spans \mathbf{T} and is l.i., and so $\{I, K\}$ is a basis

of \mathbf{T} .

Thus $\dim \mathbf{T} = 2$.