MATH 2107 - Fall 2012
Linear Algebra II
Test 1

Name: Solutions

Student ID:

Total: 25 marks (plus 1 bonus mark)
Closed book.

1. (a) [2 marks] State the definition of a subspace of a vector space.

A subset V of a vector space is a subspace if

✓(1) $v_1 + v_2 \in V \quad \forall v_1, v_2 \in V$

✓(2) $cv \in V \quad \forall c \in \mathbb{R},\ v \in V$

(b) [3 marks] Let

$$U = \{(a,\ b,\ c)^T \in \mathbb{R}^3 \mid a + b + c = 1\}.$$

Show that U is not a subspace of $\mathbb{R}^3$.

✓ $(1\ 0\ 0)^T \in U$ but $-1 \cdot (1\ 0\ 0)^T = (-1\ 0\ 0) \notin U$ so (2) fails. ✓

There are many possible counterexamples. Another one is

$(1\ 0\ 0)^T, (0\ 1\ 0)^T \in U$ but

$(1\ 0\ 0)^T + (0\ 1\ 0)^T = (1\ 1\ 0)^T \notin U$

so (1) fails

(c) [3 marks] Let P_2 denote the set of polynomials of degree at most 2, and let

$$V = \{p \in P_2 \mid p(3) = 0\}.$$

Show that V is a subspace of P_2.

Check (1) and (2):

✓(1) If $p, q \in V$, $p(3) = 0 = q(3)$ and $(p+q)(3) = 0$ so $p + q \in V$

✓(2) If $p \in V$, $p(3) = 0$ so for any $c \in \mathbb{R}$, $(cp)(3) = cp(3) = c \cdot 0 = 0$. So $cp \in V$.

✓

2. Consider the set of 2×2 matrices

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \right\}.$$

(a) [1 mark] State the definition of a linearly independent set of vectors.

✓ $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if the only solution to

$$a_1 v_1 + \ldots + a_n v_n = 0 \quad \text{is} \quad a_1 = a_2 = \ldots = a_n = 0$$

(If they use S in the definition, that is OK.)

(b) [3 marks] Determine whether the set S is linearly independent.

✓ Solve

$$a_1 \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} + a_4 \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} a_1 - a_3 + a_4 & 0 \\ a_1 + a_2 + a_3 - a_4 & -a_1 - a_3 \end{pmatrix} = 0$$

✓ $\Leftrightarrow$ $a_1 - a_3 + a_4 = 0$ $\quad$ $-a_1 - a_3 = 0$

$a_1 + a_2 + a_3 - a_4 = 0$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = 0 \quad (*)$$

✓ This has nontrivial solutions since the matrix has more columns than rows. So S is linearly dependent.

Alternatively, they could solve $(*)$ and find a nontrivial solution. (Finally they could just say the lower triangular matrices have dim 3.)

(c) [1 mark] State the definition of the span of a set of vectors.

✓ The span of $\{v_1, \ldots, v_n\}$ is the set

$$\{a_1 v_1 + \ldots + a_n v_n \mid a_1, \ldots, a_n \in \mathbb{R}\}$$

(Again using S is fine.)

(d) [2 marks] Determine whether the set S spans the vector space of 2×2 matrices $\mathbb{R}^{2\times 2}$.

✓ No. $\dim \mathbb{R}^{2\times2} = 4$ but since S is linearly dependent,

✓ $\dim \operatorname{span} S < 4$.

Alternatively, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin \operatorname{span} S$ by inspection.

3. (a) [2 marks] State the definition of a linear transformation.

A function $T: V \to W$ (V, W vector spaces) is a linear transformation if

(1) $Tv_1 + Tv_2 = T(v_1+v_2) \quad \forall v_1, v_2 \in V$

(2) $cTv = T(cv) \ \forall c \in \mathbb{R}, v \in V$.

(b) [3 marks] Let P_2 denote the vector space of polynomials of degree at most 2. Define $T : P_2 \to P_2$ by

$$T(p(x)) = p(x) + 1$$

Show that T is not a linear transformation.

Many possible counterexamples. For example,

$T(1) + T(1) = (1+1) + (1+1) = 4$

but $T(1+1) = T(2) = 2+1 = 3$. So (1) fails.

Another one is $-1\,T(1) = -1\,(1+1) = -2$

but $T(-1) = -1+1 = 0$. So (2) fails.

(c) [3 marks] Define $R : \mathbb{R}^3 \to \mathbb{R}$ by

$$R((a,\ b,\ c)^T) = a - b + c.$$

Show that R is a linear transformation.

(1) $R((a\ b\ c)^T + (a'\ b'\ c')^T) = (a-b+c) + (a'-b'+c')$

$= (a+a') - (b+b') + (c+c')$

$= R((a+a',\ b+b',\ c+c')^T)$

(2) For $s \in \mathbb{R}$,

$sR(a\ b\ c)^T = s(a-b+c) = sa - sb + sc$

$= R(sa\ sb\ sc)^T$

4. [3 marks] Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that $\{v_1, v_2, v_3\}$ is a linearly *dependent* set of vectors in V. Show that $\{Tv_1, Tv_2, Tv_3\}$ is a linearly *dependent* set of vectors in W.

Since $\{v_1, v_2, v_3\}$ are linearly dependent, there are $a_1, a_2, a_3 \in \mathbb{R}$, not all zero, such that

$a_1v_1 + a_2v_2 + a_3v_3 = 0$.

Then $0 = T(a_1v_1 + a_2v_2 + a_3v_3) = a_1Tv_1 + a_2Tv_2 + a_3Tv_3$,

so $\{Tv_1, Tv_2, Tv_3\}$ are linearly dependent.

(If they assume $v_1 = av_2 + bv_3$ without explaining why they can make that assumption, but otherwise it works, then deduct 1 mark.)