ECO 3153 Microeconomics III
Winter 2010 – Professor Shiell

Midterm I – Answers

Total marks:

1. Show that if indifference curves intersect the consumer is inconsistent.

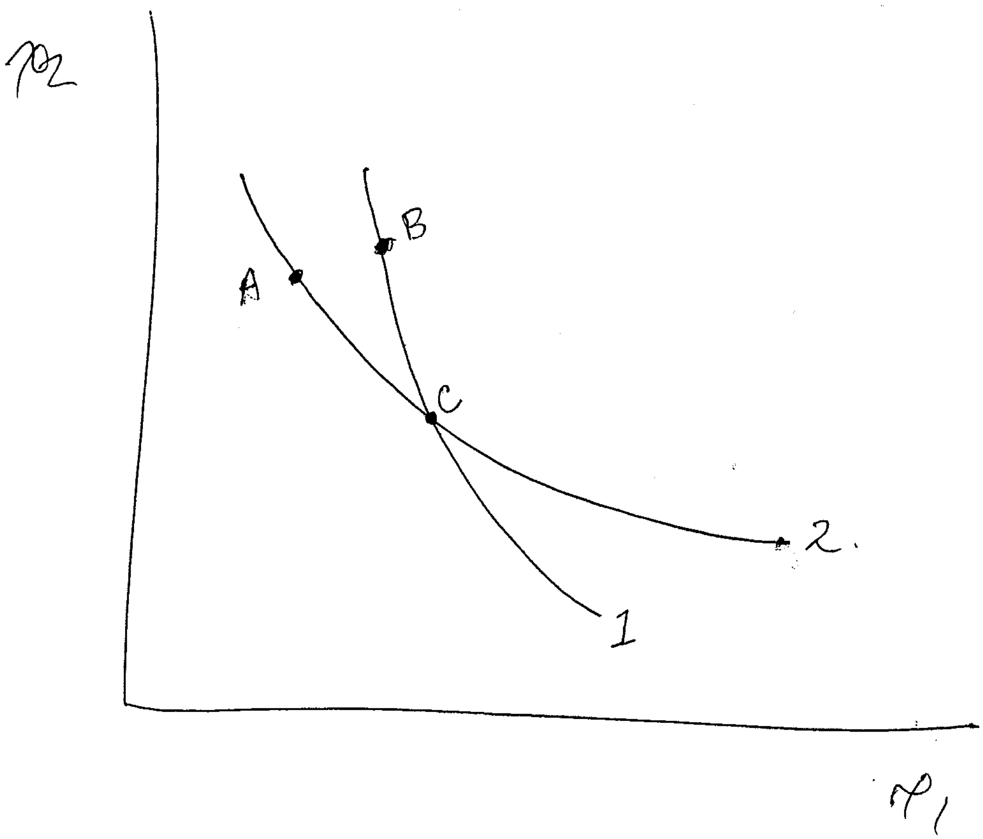
   $B \sim C$ by construction (indifference curve 1) 1

   Total: 4.

   $C \sim A$ by construction (indifference curve 2) 1

   $\Rightarrow B \sim A$ by transitivity 1

   But $B \succ A$ by non-satiation. 1



2. A consumer has the following utility function: $u(x_1, x_2) = x_1^a x_2^b$, $a + b = 1$
   a.) Derive the expenditure function, where p_1 and p_2 represent the prices of goods 1 and 2 respectively.
   b.) Obtain the indirect utility function from your answer to (a.) above.
   c.) Derive the Marshallian demand functions for goods 1 and 2 from your answer to (b.). [Hint: Use Roy's Identity.]

a.)

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad \bar{u} = x_1^a x_2^b,$$

With λ as the Lagrange multiplier, the Lagrange function is

$$\boxed{L = p_1 x_1 + p_2 x_2 + \lambda\left(\bar{u} - x_1^a x_2^b\right).} \quad 2$$

and so the first-order conditions are

Total: 11

$$\boxed{\begin{aligned} L_{x_1} &= p_1 - \lambda a x_1^{a-1} x_2^b = 0, \\ L_{x_2} &= p_2 - \lambda b x_1^a x_2^{b-1} = 0, \\ L_\lambda &= \bar{u} - x_1^a x_2^b = 0. \end{aligned}} \quad 2$$

(4.1) and (4.2) give

$$\boxed{\begin{aligned} \frac{p_1}{p_2} &= \frac{a x_2}{b x_1} \\ \Rightarrow x_2 &= \frac{p_1}{p_2}\frac{b}{a} x_1 \end{aligned}} \quad 2$$

substitute (4.4) into (4.3)

$$\boxed{\bar{u} = x_1^a \left(\frac{p_1}{p_2}\frac{b}{a} x_1\right)^b} \quad 1$$

$$= \left(\frac{p_1}{p_2}\frac{b}{a}\right)^b x_1$$

$$\boxed{\Rightarrow x_1^* = \bar{u}\left(\frac{p_1}{p_2}\right)^{-b}\left(\frac{b}{a}\right)^{-b}} \quad 1$$

and by symmetry,

$$\boxed{x_2^* = \bar{u}\left(\frac{p_2}{p_1}\right)^{-a}\left(\frac{a}{b}\right)^{-a}.} \quad 1$$

Let $m(p_1, p_2, u)$ be the expenditure function;

$$\begin{aligned} \boxed{m(p_1, p_2, u) = p_1 x_1^* + p_2 x_2^*} \quad &1/2 \\ = p_1\left[\bar{u}\left(\frac{p_1}{p_2}\right)^{-b}\left(\frac{a}{b}\right)^b\right] + p_2\left[\bar{u}\left(\frac{p_2}{p_1}\right)^{-a}\left(\frac{a}{b}\right)^{-a}\right], \quad &1/2 \\ = \bar{u} p_1^{1-b} p_2^b \left(\frac{a}{b}\right)^b + \bar{u} p_2^{1-a} p_1^a \left(\frac{a}{b}\right)^{-a} & \\ = \bar{u} p_1^a p_2^b \left(\frac{a}{b}\right)^b + \bar{u} p_2^b p_1^a \left(\frac{a}{b}\right)^{-a} & \\ \boxed{= \left[\left(\frac{a}{b}\right)^b + \left(\frac{a}{b}\right)^{-a}\right] p_1^a p_2^b \bar{u}} \quad &1 \end{aligned}$$

b.) $\boxed{\text{Invert the expenditure function to obtain the indirect utility function:}}$

$$\boxed{u^*(P_1, P_2, M) = \frac{M}{\left[\left(\frac{a}{b}\right)^b + \left(\frac{a}{b}\right)^{-a}\right] P_1^a P_2^b}} \quad 2.$$

Total: 2

Simplifying the expression in brackets:

$$\left[\left(\frac{a}{b}\right)^b + \left(\frac{a}{b}\right)^{-a}\right] = \left[\left(\frac{a}{b}\right)^b \frac{a^a}{a^a} + \left(\frac{b}{a}\right)^a \frac{b^b}{b^b}\right]$$

$$= \frac{a^{a+b} + b^{a+b}}{a^a b^b}$$

$$= \frac{a+b}{a^a b^b}$$

$$= \frac{1}{a^a b^b} \quad \text{since } a+b=1.$$

Therefore

$$u^*(P_1, P_2, M) = \left(\frac{a}{P_1}\right)^a \left(\frac{b}{P_2}\right)^b M.$$

c.) Roy's Identity: $\boxed{\dfrac{\partial u^*(P,M)}{\partial P_i} = -\lambda D_i(P,M)}$ 1

$$\Rightarrow \boxed{D_i(P,M) = -\frac{\partial u^*(P,M)/\partial P_i}{\lambda} = -\frac{\partial u^*(P,M)/\partial P_i}{\partial u^*(P,M)/\partial M}.}$$ 1

$$\boxed{\frac{\partial u^*}{\partial P_1} = -\frac{aM}{[\quad]\, P_1^{a+1} P_2^{b}}.}$$ 1

$$\frac{\partial u^*}{\partial P_2} = -\frac{bM}{[\quad]\, P_1^{a} P_2^{b+1}}$$

$$\boxed{\frac{\partial u^*}{\partial M} = \frac{1}{[\quad]\, P_1^{a} P_2^{b}}.}$$ 1

$$\therefore D_1(P,M) = \boxed{-\left\{\frac{-\dfrac{aM}{[\quad]\, P_1^{a+1} P_2^{b}}}{\dfrac{1}{[\;]\, P_1^{a} P_2^{b}}}\right\}} \tfrac{1}{2} = \boxed{a\frac{M}{P_1}} \tfrac{1}{2}.$$

$D_2(P,M) = b\dfrac{M}{P_2}$ by symmetry. 1

Total: 6.

3. a.) $u_0 = -0.1637 > u_1 = -0.1773$.

$\boxed{CV = m(p^{new}, u_0) - m(p^{new}, u_1)}$ 2.

Total: 5

$\boxed{\text{Note: } m(p^{new}, u_1) = m(p^{old}, u_0) = M = 80}$. 1/2.

$CV = m(p^{new}, u_0) - 80$.

$$\boxed{m(p^{new}, u_0) = -\frac{(\sqrt{3.12} + \sqrt{4})^2}{-0.1637}}$$ 1.

$$\boxed{= 86.65}$$ 1/2.

$\boxed{CV = 86.65 - 80 = 6.65}$ ~~5~~ 1.

b.) Since the price increase makes Bill worse off, the CV is measuring how much money we would have to pay Bill in compensation, at the new price, to make him as well off as he was before the tax. 1 1 1 3.

Total: 3

(Called WTA - willingness to accept.).

c.) $\boxed{EV = m(p^{OLD}, u_0) - m(p^{OLD}, u_1)}$ 2

Note: $m(p^{OLD}, u_0) = M = 80$. ½.

Total: 5

$EV = 80 - m(p^{OLD}, u_1)$.

$$\boxed{m(p^{OLD}, u_1) = \frac{-(\sqrt{2.62} + \sqrt{4})^2}{-0.1773}}$$ 1.

$$\boxed{= 73.86.}$$ ½

$$\boxed{EV = 80 - 73.86 = 6.14.}$$ 1

d.) EV measures the maximum amount Bill would be willing to pay to avoid having the tax policy implemented. 1 1

(Called WTP – willingness to pay).

Total: 3.

i.e. keep the old price. 1.

4 (a.) $\boxed{MRTS_{21} = \dfrac{\partial f/\partial z_1}{\partial f/\partial z_2}}$ 1

Total: 3

$$\frac{\partial f}{\partial z_1} = \frac{1}{\alpha} A \left(\quad \right)^{1/\alpha - 1} \alpha \delta_1 z_1^{\alpha-1}$$ 1

$$\frac{\partial f}{\partial z_2} = \text{etc.}$$

$$MRTS_{21} = \frac{\delta_1}{\delta_2}\left(\frac{z_1}{z_2}\right)^{\alpha-1} = \frac{\delta_1}{\delta_2}\left(\frac{z_2}{z_1}\right)^{1-\alpha}$$ 1

4(b.) $\sigma = \dfrac{d(z_2/z_1)}{d\,MRTS} \cdot \dfrac{MRTS}{z_2/z_1}$ } 2.

Total: 7

From above $MRTS = \dfrac{\delta_1}{\delta_2}\left(\dfrac{z_2}{z_1}\right)^{1-\alpha}$.

$$\Rightarrow \quad \frac{z_2}{z_1} = \left(\frac{\delta_2}{\delta_1} MRTS\right)^{\frac{1}{1-\alpha}} \Big\}\ 1$$

$$\frac{d\left(\frac{z_2}{z_1}\right)}{d\,MRTS} = \frac{1}{1-\alpha}\left(\frac{\delta_2}{\delta_1} MRTS\right)^{\frac{1}{1-\alpha}-1} \frac{\delta_2}{\delta_1} \Big\}\ 1.$$

$$\Rightarrow \quad \sigma = \frac{1}{1-\alpha}\left(\frac{\delta_2}{\delta_1} MRTS\right)^{\frac{1}{1-\alpha}-1} \frac{\delta_2}{\delta_1} \cdot \frac{MRTS}{\left(\frac{\delta_2}{\delta_1} MRTS\right)^{\frac{1}{1-\alpha}}} \quad 1.$$

1

$$= \frac{1}{1-\alpha}\left(\frac{\delta_2}{\delta_1} MRTS\right)^{-1} \frac{\delta_2}{\delta_1} MRTS.$$

$$\boxed{\sigma = \frac{1}{1-\alpha}} \quad 1.$$

4(c.) The linear form 1.

Total: 1. $f(z_1, z_2) = \delta_1 z_1 + \delta_2 z_2.$

4.(d.) From (b.), we have $\boxed{\sigma = \frac{1}{1-\alpha}.}$

Total: 1. But this expression is undefined when $\alpha = 1$. Therefore, we must evaluate the limit.

$$\boxed{\sigma = \lim_{\alpha \to 1^-} \frac{1}{1-\alpha} = +\infty.}$$ 1.

5. Show that if complements and substitutes are defined in terms of Marshallian demand derivatives, goods could be complements on the basis of the sign of $\partial D_i / \partial p_j$ and substitutes on the basis of the sign of $\partial D_j / \partial p_i$. (Hint: Use the Slutsky equation.)

Total: 10

Slutsky equation: $\dfrac{\partial D_i}{\partial p_j} = \dfrac{\partial H_i}{\partial p_j} - x_j^* \dfrac{\partial D_i}{\partial M}$ 2

1

Consider defining goods i,j as Marshallian complements; i.e. $\dfrac{\partial D_i}{\partial p_j} < 0$

1

By the Slutsky equation: $\dfrac{\partial D_i}{\partial p_j} < 0 \;\Rightarrow\; \dfrac{\partial H_i}{\partial p_j} - x_j^* \dfrac{\partial D_i}{\partial M} < 0 \;\Rightarrow\; \dfrac{\partial H_i}{\partial p_j} < x_j^* \dfrac{\partial D_i}{\partial M}$. (1)

Now by Young's Theorem $\dfrac{\partial H_i}{\partial p_j} = \dfrac{\partial H_j}{\partial p_i}$. 1 (2)

Combining (1) and (2) yields $\dfrac{\partial H_j}{\partial p_i} < x_j^* \dfrac{\partial D_i}{\partial M}$. 1 (3)

1

Now consider defining j,i as Marshallian *substitutes*; i.e. $\dfrac{\partial D_j}{\partial p_i} > 0$

By the Slutsky equation: $\dfrac{\partial D_j}{\partial p_i} > 0 \;\Rightarrow\; \dfrac{\partial H_j}{\partial p_i} > x_i^* \dfrac{\partial D_j}{\partial M}$ 1 (4)

Combining (3) and (4) $x_i^* \dfrac{\partial D_j}{\partial M} < \dfrac{\partial H_j}{\partial p_i} < x_j^* \dfrac{\partial D_i}{\partial M}$. 1

As long as $\dfrac{\partial H_j}{\partial p_i}$ is in this range, the Marshallian derivatives will have opposite signs. 1