Review Lecture

(note: The notes are subject to possible revision for improving readability; please constantly check WebCT for possible update.)

Fourier series

Fourier series (FS) of a $(2L)$-periodic piecewise continuous function $f(x)$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right],$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx.$$

When $f(x)$ is **odd**, the above FS becomes Fourier **sine** series:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

When $f(x)$ is **even**, the above FS becomes Fourier **cosine** series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}.$$

Convergence Theorem on Fourier series for periodic functions with piecewise continuous derivatives. Convergence at continuity points and jump points

A function defined on a finite interval can be extended to give a periodic function, and then the FS for the resulting function may be calculated.

Two methods of extensions, in particular:

- odd extension,
- even extension.

Chapter 11: Parametric equations and polar coordinates

Parametric equations for a plane curve:

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases}$$

where t is from a finite or infinite interval.

In the parametric equation, if y may be regarded as a function of x, **the first and second derivatives** may be calculated as follows:

- $\frac{dy}{dx} = x_t'/y_t'$ (assume the RHS is well defined)
- $\frac{d^2y}{dx^2} = \frac{dy/dx}{dx} = (dy/dx)_t'/x_t'$.

Horizontal tangent lines: $y_t' = 0$, $dy/dx = 0$

Vertical tangent lines: $x_t' = 0$, $dx/dy = 0$.

Suppose a space curve C is parametrized such that $t \in [\alpha, \beta]$ and $x(\alpha) = a$, $x(\beta) = b$.

The area below the curve (if being above the x-axis):

$$A = \int_a^b y dx = \int_\alpha^\beta y(t) x_t' dt$$

where $x(\alpha) = a$ and $x(\beta) = b$.

Arc length:

$$L = \int_\alpha^\beta \sqrt{(x_t')^2 + (y_t')^2} dt.$$

Area of a surface obtained by rotating a curve about x-axis:

$$S = \int_\alpha^\beta y(t) \sqrt{(x_t')^2 + (y_t')^2} dt.$$

Polar coordinates: Each point may be represented by a pair (r, θ).

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Area (of a sector):

$$A = (1/2)\int_a^b r^2(\theta)d\theta.$$

This is the area bounded by the curve $r(\theta)$, $\theta \in [a,b]$ and and two lines $\theta = a$, $\theta = b$. Do not forget the factor $(1/2)$. An easy method to correctly remember this formula is to test it with a unit circle.

Arc length:

$$L = \int_a^b \sqrt{r^2 + (r'_\theta)^2}d\theta.$$

Chapter 13: Vectors and geometry of space

Inner product: for both 2-D and 3-D.

Cross product (only for 3-D):

$$\mathbf{a} = (a_1, a_2, a_3), \quad \mathbf{b} = (b_1, b_2, b_3). \tag{1}$$

Then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Projection of one vector onto another one.

Equations for lines and planes.

Distance D between

- A point (x_0, y_0, z_0) and a plane: $D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$
- A point and a line

Classification of quadratic surfaces.

Chapter 14: Vector functions

Consider a curve given by the vector equation $r = r(t)$ with $t \in [a,b]$.

Arc length:

$$L = \int_a^b |r'(t)|dt = \int_a^b \sqrt{(x_t')^2 + (y_t')^2 + (z_t')^2}dt$$

Unit tangent vector:

$$T(t) = r'(t)/|r'(t)|,$$

where $|r'(t)|$ is the length of the vector $r'(t)$.

Curvature:

$$\kappa(t) = |\frac{dT}{ds}| = \frac{|r' \times r''|}{|r'|^3}.$$

Unit tangent, unit normal vector, binormal vector, osculating plane

$$N = T'/|T'|, \qquad B = T \times N.$$

Motion

Position: $r(t)$, velocity: $r'(t)$, speed: $|r'(t)|$, acceleration: $r''(t)$.

Chapter 15: Partial derivatives

Tangent lines and tangent planes:

2-D: For the curve implicitly defined by the equation $F(x,y) = 0$, the equation of the tangent line at (x_0, y_0) is

$$y = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}(x - x_0) + y_0.$$

3-D: Equation of the tangent plane of a surface at the point (x_0, y_0, z_0):
When the surface is explicitly given by $z = f(x,y)$:

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0).$$

When the surface is defined by $F(x,y,z) = 0$:

$$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0.$$

Differentials:

$$z = f(x,y), \qquad dz = f_x dx + f_y dy$$

$$w = f(x,y,z) \qquad dw = f_x dx + f_y dy + f_z dz.$$

The differential may be used to give linear approximation of a given function.

The chain rule:

Case 1: $x = f(t)$, $y = g(t)$, and $z = F(x,y) = G(t)$ (so z may be ultimately expressed as a function $G(t)$ of t). Then

$$z_t = F_x x_t + F_y y_t = F_x f_t + F_y g_t.$$

Case 2: $x = f(s,t)$, $y = g(s,t)$, and $z = F(x,y) = G(s,t)$. Then

$$z_s = F_x x_s + F_y y_s, \qquad z_t = F_x x_t + F_y y_t.$$

The gradient vector.
Two variable function $z = f(x,y)$, $\nabla f = (f_x, f_y)$.
Three variable function: $w = f(x,y,z)$, $\nabla f = (f_x, f_y, f_z)$.

Directional derivative: Let u be a unit vector.

$$D_u = \nabla f \cdot u = |\nabla f| \cos\theta$$

where θ is the angle between f and u. When u is in the direction of the gradient, the directional derivative has the largest magnitude.

Critical points of a function (of two variables).

Second derivatives test for local minimum/maximum and saddle points.

Lagrange multipliers: Given a function $f(x,y,z)$ subject to constraints:

$$g(x,y,z) = c, \tag{2}$$

where c is a constant. Then the method of Lagrange multipliers may be used to find the constrained maximum and minimum if existing.
The following steps may be used to find the extreme values.

- Define the function $F = f - \lambda(g - c)$.
- Give the derivative condition: $F_x = 0$, $F_y = 0$ and $F_z = 0$.
- Next, the following equation system is solved

  $$\begin{cases} f_x = \lambda g_x, \\ f_y = \lambda g_y, \\ f_z = \lambda g_z, \\ g = c \end{cases}$$

  Note: the equality constraint $g = c$ should be included in the above equation system.
- Solve the above equation system to get several solutions. Evaluate f at theses solutions.

Such constrained minimization/maximization problems may appear in other very similar forms and the method of Lagrange multipliers is still applicable:

- Find the extreme values of a three variable function $f(x,y,z)$ subject to two equality constraints $g(x,y,z) = c_1$ and $h(x,y,z) = c_2$.
- Find the extreme values of a two variable function $f(x,y)$ subject to one constraint $g(x,y) = k$.

Chapter 16: Multiple integrals

Fubini's Theorem.

Rectangular regions: If $f(x,y)$ is continuous on the rectangle $S = \{(x,y) | a \leq x \leq b,\ c \leq y \leq d\}$, then

$$\iint\limits_S f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

Type I region:

$$\iint\limits_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dydx.$$

Type II region:

$$\iint\limits_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} dxdy.$$

Highlight: There exist regions which are of both type I and type II. In this case, we may evaluate the double integral by converting it into an iterated integral of either type I or type II; also, the iterated integral of type I (resp., II) may be converted to the iterated integral of type II (resp. I).

Double integral in polar coordinates:

$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Triple integrals.

Type I:

$$\iint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$

Thus, the region E is bounded by an upper surface and a lower surface and a cylinder whose projection onto the $x-y$ plane is D.
Depending on the structure of D, the above double integral may be further evaluated by iterated integrals.

Type II:

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dA.$$

Type III:

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dA.$$

Triple integral in cylindrical coordinates:

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(x,y)}^{u_2(x,y)} f(r\cos\theta, r\sin\theta) r dz dr d\theta.$$

In evaluating the integral, we further write $u_i(x,y) = u_i(r\cos\theta, r\sin\theta)$ for $i = 1,2$.

Triple integrals in spherical coordinates:

$$\iiint_E f(x,y,z)dV$$
$$= \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi.$$

Chapter 17: Vector calculus

Line integrals.

Type I.

$$\text{2-D}: \quad \int_C f(x,y)ds = \int_a^b f(u(t),v(t))\sqrt{(x_t')^2+(y_t')^2}dt$$
$$\text{3-D}: \quad \int_C f(x,y,z)ds = \int_a^b f(u(t),v(t),w(t))\sqrt{(x_t')^2+(y_t')^2+(z_t')^2}dt.$$

Type II:

$$\begin{aligned}
\text{2-D}: \quad \int_C F\cdot dr &= \int_C P(x,y)dx+Q(x,y)dy \\
&= \int_a^b [P(u(t),v(t))u'(t)+Q(u(t),v(t))v'(t)]dt. \\
\text{3-D}: \quad \int_C F\cdot dr &= \int_C P(x,y)dx+Q(x,y)dy \\
= \int_C [P(u(t),v(t),w(t))u'(t) &+ Q(u(t),v(t),w(t))v'(t) \\
&+ R(u(t),v(t),w(t))w'(t)]dt
\end{aligned}$$

Fundamental theorem:

$$\int_C F\cdot dr = f(r(b)) - f(r(a))$$

where F is conservative and f is its potential function.

The sufficient and necessary condition:
A 2-D vector field $F = P(x,y)i + Q(x,y)j$ with continuous partial derivatives is conservative if and only if

$$P_y = Q_x.$$

This criterion may be used to **find the potential function** (check notes) for a given conservative vector field.