



# Université d'Ottawa - University of Ottawa

Faculté des sciences  
Mathématiques et de statistique

Faculty of Science  
Mathematics and Statistics

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## MAT 1302 D – FINAL EXAM – April, 12th 2011

Instructor: Termeh Kousha

NAME \_\_\_\_\_

I.D.# \_\_\_\_\_

Duration: 3 hours

**NO CALCULATORS. NO BOOKS. NO NOTES.**

**Instructions:** This exam consists of 10 multiple choice questions and 8 Long answer questions on 16 pages. The marks for each question are as listed with the question itself. The total value of the exam is 55 points.

Place your answers to the multiple choice questions in the boxes below. You may use the back of pages for short and long answer questions.

### Answers to Multiple Choice:

B

#1

D

#2

A

#3

B

#4

B

#5

A

#6

B

#7

A

#8

B

#9

A

#10

Multiple Choice Questions, each is worth 2 marks, 20 marks overall.

1. For which values of  $h$  and  $k$  does the system

$$\begin{aligned}x_1 + 3x_2 &= k \\4x_1 + hx_2 &= 8\end{aligned}$$

have no solutions.

(a)  $h = 12, k = 2.$

()  $h = 12, k \neq 2.$

(c)  $k \neq 2$  and  $h \in \mathbb{R}.$

(d) For all  $h, k \in \mathbb{R}.$

(e) For all values of  $k$  and  $h$  the system has solutions.

$$\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & k \\ 0 & -12+h & -4k+8 \end{bmatrix}$$

$\rightarrow$  the system is inconsistent if  $h=12$  and  $k \neq 2$

2. Let  $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 4 & 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ,  $A^T B$  is

(a) Not defined.

(b)  $\begin{bmatrix} 11 \\ 7 \\ 4 \end{bmatrix}.$

(c)  $\begin{bmatrix} 4 \\ 7 \\ 11 \end{bmatrix}.$

()  $\begin{bmatrix} 15 \\ 2 \\ 6 \end{bmatrix}.$

(e)  $\begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$

$$A^T B = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 2 \\ 6 \end{bmatrix}$$

3. Let  $A$  and  $B$  be square matrices of size  $n$ . Which of the following statements are true?

1.  $\det(A) = 0$  if and only if the equation  $Ax = 0$  has ONLY trivial solution.
2.  $A$  is invertible if and only if  $A$  has eigenvalue 0.
3.  $\det(AB) = \det(BA)$ . ✓
4. If  $\text{Col}A = \mathbb{R}^n$ , then  $\dim \text{Nul}A = 0$ . ✓
5. If  $\lambda$  is an eigenvalue of  $A$ , then  $\lambda$  is also an eigenvalue of  $A^T$ . ✓

- (a) Statements 3,4,5      (b) Statements 1,2,4      (c) Statements 2,3,4,5  
 (d) Statements 1,4,5      (e) All of the statements are False!

4. Let  $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ,  $\vec{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ ,  $\vec{u}_3 = \begin{bmatrix} -2a \\ b \\ -a \end{bmatrix}$  be vectors in  $\mathbb{R}^3$ . Under which condition  $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$  is a basis for  $\mathbb{R}^3$ ?

(a)  $a = 0, b \in \mathbb{R}$ .

(b)  $a \neq b$ .

(c)  $a \neq -b$ .

(d)  $a = -b$ .

(e)  $a = b$ .

5. Let  $A = PDP^{-1}$ , where  $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$  and  $D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ . Compute  $A^3$ .

(a)  $\begin{bmatrix} 0 & -8 \\ -8 & 0 \end{bmatrix}$

(b)  $\begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 8 & -8 \\ 0 & 4 \end{bmatrix}$

(e)  $\begin{bmatrix} 0 & 16 \\ 0 & 0 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \rightarrow P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}$$

$$A^3 = P D^3 P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 16 \\ 16 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix}$$

6. Suppose  $A$ ,  $B$  and  $C$  are three  $3 \times 3$  matrices with  $\det A = -1$ ,  $\det B = 2$ ,  $\det C = 3$ . Find

$$\det(2A^5 C^T B^T C^{-1})$$

(a) -16.

(b) 16.

(c) -4.

(d) 4.

(e) -36.

$$= 2^3 \det(A^5) \det(C^T) \det(B^T) \det(C^{-1})$$

$$2^3 (-1) (3) (2) \left(\frac{1}{3}\right) = -16$$

7. Let  $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 2 & 4 \end{bmatrix}$ , then

~~(a)~~  $B$  has 3 distinct eigenvalues and  $\det B = 16$

(b)  $B$  has 2 distinct eigenvalues and  $\det B = 16$

(c)  $B$  has 3 distinct eigenvalues and  $\det B = 9$

(d)  $B$  has 1 distinct eigenvalue and  $\det B = 16$

(e)  $B$  has 2 distinct eigenvalues and  $\det B = 9$

8. Which of the following statements are true?

1. It is possible for a set of 4 vectors in  $\mathbb{R}^3$  to be linearly independent.

2. Any basis of  $\mathbb{R}^5$  must consist of 5 vectors. ✓

3. It is possible for a set of 4 vectors to span  $\mathbb{R}^3$ . ✓

4. Any basis of a subspace contains the zero vector.

5. A subspace can have more than one basis. ✓

(a) Statements 2,3,5      (b) Statements 1,2,4,5      (c) Statements 2,3,4,5

(d) Statements 1,4,5      (e) All of the statements are False!

9. Which sets are linearly independent?

$$A = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix} \right\}, B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

$$C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} \right\}, D = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

- (a) A and B.  
 (b) C and D.  
 (c) A and C.  
 (d) All of them.  
 (e) None of them.

10. Which of the following statements are true for an  $n \times n$  matrix  $A$  ?

1.  $\det A = \det A^T$  ✓  
 2. For every  $\vec{b} \in \mathbb{R}^n$ , the equation  $A\vec{x} = \vec{b}$  has a solution.  
 3. The equation  $A\vec{x} = 0$  always has a solution. ✓  
 4.  $\dim(\text{Nul}A) \leq n$ . ✓  
 5.  $\text{Rank}A = n$ .

- (a) Statements 1,3,4      (b) Statements 3,4,5,      (c) Statements 1,2,4  
 (d) Statements 1,4,5      (e) All of the statements are True

Long Answer Questions:

1. [3 marks] Consider the production model  $x = Cx + d$  for an economy with two sectors, where

$$C = \begin{bmatrix} 0 & .5 \\ .6 & 0.2 \end{bmatrix} \quad d = \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

Determine the production level necessary to satisfy the final demand.

$$(I - C)\vec{x} = \vec{d}$$

$$\vec{x} = (I - C)^{-1}\vec{d}$$

$$(I - C)^{-1} = \left( \begin{bmatrix} 1 & -0.5 \\ -0.6 & 0.8 \end{bmatrix} \right)^{-1} = 2 \begin{bmatrix} 0.8 & 0.5 \\ 0.6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.6 & 1 \\ 1.2 & 2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1.6 & 1 \\ 1.2 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 52 \end{bmatrix}$$

2. [3 marks] Let  $z = 2 - 2i$ ,  $w = -1 + i$  and  $r = -2$ . Calculate the following.

a.  $\frac{\bar{z}}{w} = \frac{2 + 2i}{-1 + i} \times \frac{-1 - i}{-1 - i} = \frac{-4i}{2} = -2i$

2 points

b.  $|rz| = |r||z| = 2 \sqrt{4+4} = 2\sqrt{8} = 4\sqrt{2}$

1 point

3. [5 marks] Let  $A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$ .

(a) Find the solution set of the homogeneous system  $A\vec{x} = 0$  in parametric vector form. What is the geometric description?

(b) Describe the solution of  $A\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$  in the parametric vector form. Provide a geometric solution comparison with the solution set in part (a).

(a) 
$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \dots \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3$  free.

$$x_1 - 5x_3 = 0$$

$$x_2 + 2x_3 = 0$$

$$x_3 = t \quad t \in \mathbb{R}$$

$$t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

Solution is  $\text{span}\{\vec{u}\}$

2 points

0.5 point



Its a line passes through origin.

(b) 
$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \sim \dots \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free.

$$x_1 = -2 + 5t$$

$$x_2 = 1 - 2t$$

$$x_3 = t$$

$$t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\vec{u}$

The solution set is a line through the origin parallel to the line  $\vec{u}$ .

2 points

**Question 3, continue**

4. [7 marks] Let  $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ .

- Find the characteristic polynomial of  $A$ . List the eigenvalues and their multiplicities.
- Find all the corresponding eigenvectors.
- If possible, diagonalize the matrix  $A$ . That is, find a diagonal matrix  $D$  and invertible matrix  $P$  such that  $A = PDP^{-1}$ . (Note: You are not asked to find  $P^{-1}$ .)

(a) 
$$\begin{vmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda) \begin{vmatrix} 4-\lambda & 0 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda)^2 (4-\lambda) = 0$$

$$\lambda = 4 \leftarrow \text{mul } 1$$

$$\lambda = 5 \leftarrow \text{mul } 2$$

2 points b) For  $\lambda = 4$

$$[A - \lambda I : 0] = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2$  free

$x_1 = -\frac{1}{2}t$   
 $x_2 = t$   
 $x_3 = 0$

$t \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \rightarrow \text{take}$

$v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

2 points For  $\lambda = 5$

$$[A - 5I : 0] \sim \dots \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2, x_3$  free

$x_1 = -2x_3$   
 $x_2 = t$   
 $x_3 = s$

$\begin{bmatrix} -2s \\ t \\ s \end{bmatrix} =$

1 point

c).  $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$P = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$5 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Question 4, continue

5. [3 marks] Is the matrix  $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$  invertible? If so, find  $A^{-1}$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array} \right] \sim$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array} \right]$$

$\underbrace{\hspace{10em}}_{I} \qquad \underbrace{\hspace{10em}}_{A^{-1}}$

6. [3 points] Find the determinant of  $A = \begin{bmatrix} 2 & -4 & 6 & 1 \\ 1 & 4 & 0 & 2 \\ 2 & 3 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$ .

$$|A| = (-1) \begin{vmatrix} 1 & 4 & 0 \\ 2 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -4 & 6 \\ 2 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= (-1) \begin{vmatrix} 1 & 4 & 0 \\ 0 & -5 & 2 \\ 0 & -5 & 1 \end{vmatrix} + 2(2) \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\underbrace{(-1) [-5 + 10]}_{-5} + 4 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 7 & -4 \\ 0 & 1 & -2 \end{vmatrix}$$

$$4 \left( \underbrace{-14 + 4}_{-10} \right)$$

$$\underbrace{\quad}_{-40}$$

$$= -45$$

7. [6 marks] Consider the matrix  $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}$ , whose reduced echelon form

is  $\begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -9 & 0 & 5 & 0 \\ 0 & \textcircled{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

- a. Find a bases for  $\text{Col}A$ . What is the  $\text{Rank}A$ ? free
- b. Find a bases for  $\text{Nul}A$ . What is the dimension  $\text{Nul}A$ ?
- c. State the equation relating the number of columns of a matrix, rank and the dimension of the null space.

a) columns 1, 2, 4 have pivots.

2 points basis for  $\text{COL} A = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix} \right\}$

$$\text{Dim COL}(A) = \text{Rank } A = 3$$

3 points

b) N/W/A From RREF:  $x_3, x_5$  are free

$$\begin{aligned} x_1 - 9x_3 + 5x_5 &= 0 \\ x_2 + 2x_3 - 3x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}$$

$$t \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

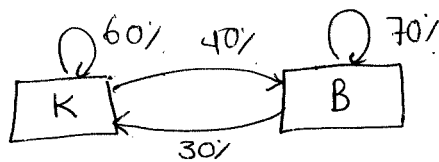
$$x_3 = t$$

$$x_5 = s$$

Basis for  $\text{Nul}(A) = \left\{ \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

$$\text{Dim Nul } A = 2.$$

c).  $\text{Dim Nul } A + \text{Rank } A =$   
 $2 + 3 = 5$  ← number of columns.



8. [5 marks] There are 20 cockroaches living in a suite of a five star hotel. The cockroaches move between the kitchenette and bathroom. At the beginning, there are 10 cockroaches in the kitchenette and 10 in the bathroom. Every week 40% of the cockroaches in the kitchenette go to the bathroom and 30% of the cockroaches in the bathroom go to the kitchen.

$$\vec{x}_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \rightarrow \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

a. Determine the migration matrix associated with this model.

1 point

$$P = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

b. Compute the expected number of cockroaches in the bathroom after 2 weeks.

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 6+3 \\ 4+7 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \end{bmatrix} \text{ after a week}$$

2 points

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 9 \\ 11 \end{bmatrix} = \begin{bmatrix} 5.4+3.3 \\ 3.6+7.7 \end{bmatrix} = \begin{bmatrix} 8.7 \\ 11.3 \end{bmatrix} \text{ after two weeks}$$

$$\begin{bmatrix} 8.7/20 \\ 11.3/20 \end{bmatrix} \rightarrow$$

c. After the migration goes on for a long period of time, find the proportion of the cockroaches in the kitchenette and bathroom. (Hint: Find the steady-state vector of the migration matrix.)

2 points

$$P\vec{q} = \vec{q} \Rightarrow (P-I)\vec{q} = \vec{0}$$

$$\Rightarrow P-I = \begin{bmatrix} -0.4 & 0.3 \\ 0.4 & -0.3 \end{bmatrix}$$

$$[P-I : 0] \sim \begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{3}{4}x_2 \quad \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix} \rightarrow \text{Probability vector} \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

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**Extra Page 2. Do NOT detach this page!**