## Part A: Multiple Choice Questions (2 marks each)

**A1.** In polar coordinates, the equation of the circle $(x-1)^2 + y^2 = 1$ is
(a) $r = \theta$, (b) $r = 1$, (c) $r = 1 + \sin\theta$, (d) $r = 2\cos\theta$, (e) $r^2 = 1 + 2r\sin\theta$.

**A2.** The area enclosed by one loop of the lemniscate $r^2 = \sin 2\theta$ $(0 \le \theta \le \frac{\pi}{2})$ is
(a) $\frac{1}{4}$, (b) $\frac{3}{4}$, (c) $\frac{1}{2}$, (d) $\frac{3}{5}$, (e) 1.

**A3.** Suppose that the angle between the vectors $\mathbf{u} = \mathbf{i} - \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + a\mathbf{j}$ is $\frac{\pi}{3}$. Then $a$ is equal to
(a) $\pm\sqrt{3}$, (b) $\frac{\pm\sqrt{3}}{2}$. (c) $\pm 3$, (d) $\pm 2$, (e) $\pm 3$.

**A4.** The acceleration vector $\mathbf{r}''(t)$ of $\mathbf{r}(t) = t^3\ \mathbf{i} + (\sin \pi t)\mathbf{j}$ at $t = \frac{1}{3}$ is
(a) $2\mathbf{i} - \frac{\sqrt{3}}{2}\pi^2\mathbf{j}$, (b) $2\mathbf{i} + \frac{\pi^2}{2}\mathbf{j}$, (c) $3\mathbf{i} + \frac{\pi^2}{2}\mathbf{j}$, (d) $3\mathbf{i} - \frac{1}{2}\mathbf{j}$, (e) None of the above.

**A5.** Let $f(x, y) = x^3y^2$. Then $f_{xx} + f_{xy}$ is equal to
(a) $12x^2y$, (b) $3x^2 + 2y$, (c) $x^2 + y$, (d) $6x^2y + 6xy^2$, (e) $6xy + 3x^2y$.

**A6.** The tangent plane to the surface $x^2 + y^2 - 2z^2 = 3$ at the point $(2, 1, 1)$ can be written as
(a) $-2(x-2) + 2(y-1) - (z-1) = 0$, (b) $2(x-2) + (y-1) - 4(z-1) = 0$.
(c) $(x-1) + 9y - 1) + (z-1) = 0$, (d) $2x + y + z = 3$, (e) $2(x-2) + (y-1) - 2(z-1) = 0$.

**A7.** Suppose that the gradient of $f(x, y)$ is given by $\nabla f = y\mathbf{i} + x\mathbf{j}$ and suppose that $w = f(s+t, s-t)$. Then $\frac{\partial w}{\partial t}$ is equal to
(a) $s^2t$, (b) $s - t$, (c) $-2t$, (d) $2s$, (e) $s + t$.

**A8.** Suppose $w = f(x^4 + y^2)$. Applying the chain rule twice, we have $\frac{\partial^2 w}{\partial x \partial y} =$
(a) $8x^3y\ f''(x^4+y^2)$, (b) $(24x^2y + 8x^3)f'(x^4+y^2)$, (c) $24x^2y\ f''(x^4+y^2)$,
(d) $(12x^2+2)f''(x^4+y^2)$, (e) $(4x^3+2y)f''(x^4+y^2)$.

**A9.** In spherical coordinates, the volume element $dV \equiv dxdydz$ is given by
(a) $\rho^2\cos\phi\ d\rho d\phi d\theta$, (b) $\rho^2\sin\theta\cos\phi\ d\rho d\phi d\theta$ (c) $\rho^2\cot\phi\sin\theta\ d\rho d\phi d\theta$,
(d) $\rho^3\cos\theta\ d\rho d\phi d\theta$, (e) $\rho^2\sin\phi\ d\rho d\phi d\theta$.

**A10.** Let $C$ be an arc whose parametric form is $x(t) = 1,\ y(t) = t,\ z(t) = t^2$ with $0 \le t \le 1$. Then the line integral $\int_C y\ dx + z\ dy + x^2y\ dz$ is equal to
(a) 0, (b) 1, (c) 2, (d) $-1$, (e) None of the above.

**A11.** Consider the function of period $2\pi$ defined on the interval $[-\pi, \pi]$ by

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t \le \pi, \\ 0 & \text{if } -\pi < t < 0. \end{cases}$$

Then the Fourier expansion of $f$ is
(a) $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{2n}$, (b) $\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nt}{n}$, (c) $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{2n-1}$,

(d) $\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nt}{n}$, (e) None of the above.

**A12.** From the Fourier expansion $t \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nt$ with $-\pi < t < \pi$ and the identity $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, we have the following Fourier expansion for $t^2$ with $-\pi < t < \pi$:

(a) $t^2 \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t \sin nt$, (b) $t^2 \sim \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos nt$,

(c) $t^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^2} \cos nt$, (d) $t^2 \sim 4\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^2} \cos nt$,

(e) None of the above.

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Part B: Do All Questions (6 Marks Each)

**B1.** Consider a curve defined by the parametric equations $x = e^t \cos t$, $y = e^t \sin t$, or, putting this in vector form, $\mathbf{r}(t) = e^t(\cos t\ \mathbf{i} + \sin t\ \mathbf{j}) \equiv e^t(\cos t,\ \sin t)$.

(a) Find the velocity vector $\mathbf{r}'(t)$, the acceleration vector $\mathbf{r}''(t)$ and the speed $v(t) \equiv |\mathbf{r}'(t)|$ at time $t$.

(b) Find the arc length of the portion of this curve between $t = 0$ and $t = 1$.

**B2.** Set up three equations for finding the closest point on the cubic curve $y^2 + y = x^3 + x^2 - 1$ to the origin $(0, 0)$ by using a Lagrange multiplier $\lambda$, and eliminate $\lambda$ to reduce them to two equations. (The final answer for the closest point is not required; -do not try to find it.) Use the square of the distance between a point $(x, y)$ on the curve and the origin as your objective function. Clearly indicate your constraint function.

**B3.** Locate all critical points of the function $f(x, y) = x^3y - xy + x^2$ and classify each of them (local maximum, local minimum or saddle point).

**B4.** Find the value of the double integral

$$\int\!\!\int_D \frac{x^2}{1-y}\, dA,$$

where $D$ is the region enclosed by the line $y = x$ and the cubic curve $y = x^3$, between $x = 0$ and $x = 1$.

**B5.** Use the polar coordinates to compute the moment of inertia

$$I_y = \int\!\!\int_P y^2\, dxdy,$$

where $P$ is the semi-circular plate which is bounded by the $y$-axis and the semi-circle $x = \sqrt{2 - y^2}$.

**B6.** Show that the vector field $\mathbf{F} = (2\sin x + y^2)\mathbf{i} + (2xy + \cos y)\mathbf{j}$ is conservative, (that is, $\mathbf{F}$ is a gradient field.) Then find a potential function for this vector field.