1) a) $Var\{aX\} = E\{(aX - E\{aX\})^2\} = E\{(aX - a\mu)^2\}$

$= E\{a^2X^2 + a^2\mu^2 - 2a^2\mu E\{X\}\} = a^2 E\{X^2\} - a^2\mu^2$

b) $(X-\mu)^2 \geq 0 \Rightarrow E\{(X-\mu)^2\} \geq 0 \Rightarrow Var\{X\} \geq 0$

c) Showed in Tutorial 2 $\quad cov\{X,Y\} = E\{XY\} - \mu_x \mu_y$

$X \perp Y \Rightarrow E\{XY\} = \mu_x \mu_y \Rightarrow cov\{X,Y\} = 0$

d) $Var\{aX+b\} = E\{(aX+b-(a\mu_x+b))^2\}$

$= E\{a^2X^2 + a^2\mu_x^2 - 2a^2\mu_x X\}$

$= a^2 E\{(X-\mu_x)^2\} = a^2 Var\{X\}$

2)

a) $$Y = \begin{cases} (-3)^2+2(-3)+2 = 5 & w.p.\ .2 \\ (-1)^2+2(-1)+2 = 1 & w.p.\ .3 \\ (1)^2+2(1)+2 = 5 & w.p.\ .1 \\ 3^2+2(3)+2 = 17 & w.p.\ .4 \end{cases} = \begin{cases} 1 & w.p.\ .3 \\ 5 & w.p.\ .3 \\ 17 & w.p.\ .4 \end{cases}$$

b) $E\{Y\} = .3 \times 1 + .3 \times 5 + .4 \times 17 = 8.6$

$E_Y\{Y\} = E_X\{X^2+2X+2\} = 5 \times .2 + 1 \times .3 + 5 \times .1 + 17 \times .4$

$= 8.6$

$Var\{Y\} = E\{Y^2\} - E^2\{Y\}$, $\quad E\{Y^2\} = 1^2 \times .3 + 25 \times .3 + 289 \times .4$

$= 123.4$

$= 123.4 - (8.6)^2 = 123.4 - 73.96 = 49.44$

c) $p_X = \sum x_m^2 f_X(x_m) = (-3)^2 \times .2 + (-1)^2 \times .3 + 1^2 \times .1$

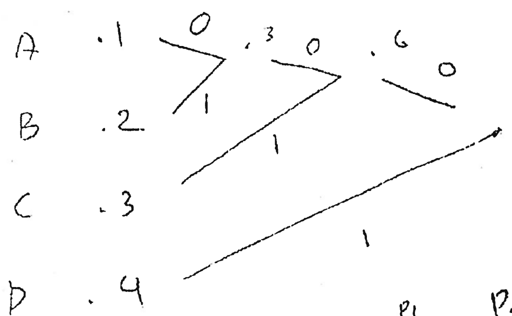
$+ 3^2 \times .4 = 9 \times .6 + 1 \times .4$

$= 5.8$

d, e) MatLab Code

3. MatLab

4. $H(X) = \sum_{i=1}^{4} - P_i \log P_i = -(.2 \log .2 + .3 \log .3 + .4 \log .4 + .1 \log .1)$

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$X = \{ \overset{P_1}{-5}, \overset{P_2}{-1}, \overset{P_3}{1}, \overset{P_4}{5} \}$ $\quad Y = \{ \overset{P_2+P_3}{1}, \overset{P_1+P_4}{25} \}$

5) $H(X) = -P_1 \log P_1 - P_2 \log P_2 - P_3 \log P_3 - P_4 \log P_4$

$H(Y) = -(P_1+P_4) \log (P_1+P_4) - (P_2+P_3) \log (P_2+P_3)$

$= -P_1 \log (P_1+P_4) - P_2 \log (P_2+P_3) - P_3 \log (P_2+P_3)$
$- P_4 \log (P_1+P_4)$

$\log X$ is increasing func. $\Rightarrow$

$\log (P_1+P_4) \geq \log P_1$
$\geq \log P_4$ $\Rightarrow$ $-\log (P_1+P_4) \leq -\log P_1$ and $-\log P_4$

$-\log (P_2+P_3) \leq -\log P_2$
$-\log P_3$

$\Rightarrow H(X) \geq H(Y)$

b/ $H(X) = H(Y) \longrightarrow$ for $\underline{Y = 5X}$

6.) $H(X) = -\sum p_i \log p_i = .725$

bw = 6000, guard = 2000, Sample_freq = 14000

$H(X) = 2.4087$

H_bps = 3,03,722 bit/second

7) $\pi(\infty) = A' \pi(\infty)$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} .6 & .1 & .1 \\ .10 & .9 & .1 \\ .3 & 0 & .8 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

$$\begin{cases} \pi_1 = .6\pi_1 + .1\pi_2 + .1\pi_3 \\ \pi_2 = .1\pi_1 + .9\pi_2 + .1\pi_3 \Rightarrow \pi_2 = 2.5\pi_1 \\ \pi_3 = .3\pi_1 + .8\pi_3 \Rightarrow .2\pi_3 = 3\pi_1 \Rightarrow \pi_3 = 1.5\pi_1 \end{cases}$$

$\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \pi_1 + 2.5\pi_1 + 1.5\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{5}$

$$\Rightarrow \begin{cases} \pi_1 = 1/5 \\ \pi_2 = 1/2 \\ \pi_3 = 3/10 \end{cases}$$

$H(X) = \pi_1 H_1(X) + \pi_2 H_2(X) + \pi_3 H_3(X)$

8) $t_N =$ waiting time for $N$ consecutive tails

$$t_{N+1} = (t_N+1)p + (t_N+1+t_{N+1})(1-p)$$

$$t_0 = 0$$

$$\Rightarrow t_1 = p + (0+1+t_1)(1-p)$$

$$t_1 = p + t_1(1-p) + (1-p)$$

$$t_1 p = 1 \Longrightarrow t_1 = 1/p$$

$$t_{N+1} = t_N p + p + t_N(1-p) + (1-p) + t_{N+1}(1-p)$$

$$t_{N+1} \cdot p = t_N + 1 \Rightarrow t_{N+1} = \frac{t_N+1}{p}$$

assume $t_N = \alpha p^{-N} + C$

$$\Rightarrow t_{N+1} = \alpha p^{-(N+1)} + C \quad , \quad p t_{N+1} = t_N + 1$$

$$\Rightarrow p(\alpha p^{-(N+1)} + C) = \alpha p^{-N} + C + 1$$

$$\Rightarrow \cancel{\alpha p^{-N}} + pC = \cancel{\alpha p^{-N}} + C + 1 \Rightarrow C = -\frac{1}{1-p}$$

& we know $t_1 = \frac{1}{p} \Rightarrow \alpha = \frac{1}{1-p}$

$$\Rightarrow t_N = \frac{1}{1-p}\left(p^{-N} - 1\right)$$