Systems of Linear Equations

We've seen these before. Almost every problem we solved became one of these. The general form of a linear system is

$$\begin{array}{ccccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n-1}x_{n-1} & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n-1}x_{n-1} & + & a_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n-1}x_{n-1} & + & a_{m,n}x_n & = & b_m \end{array}$$

notice that there are n variables and m equations. As before, the objective is to find values of x_1, x_2, ... , x_n such that EACH AND EVERY equation is satisfied, all at the same time. There are three different numbers of solutions you can get. You can have one, unique, solution. You can find an infinite number of solutions. Finally, there can be NO solution to a given linear system.

Definition: A system of equations that has NO solutions is called *inconsistent*.

An equation that has any number of solutions (other than zero) is simply consistent. Beyond the one type of result, inconsistency and NO answers, you can have either a unique solution (ONE answer) or an infinite set of them.

Definition: The *General Solution* of a system is a set of all solutions of the system. It has to include ALL solutions. It can be empty (if the system has no solutions) or it can be just one element. It's usually best written in parametric form (it can be a line, a plane, or bigger, but we've seen those).

Definition: A *Homogeneous* system has ONLY zeros on the right hand side as written above. So, NO constant terms.

Property: All homogeneous systems are consistent.

Example:
Here's a system of equations:

$$\begin{aligned} 3x + 2y - z &= 2, \\ x - 4y + 4z &= -1, \\ -2x + z &= 0. \end{aligned}$$

Where could this have come from? What would the answer mean?

Here's one: it could be simply finding the intersection (or finding the lack of intersection) of three planes in $\mathbb{R}^3$, each one represented by an equation.

Here's another: it could be the result of a spanning question:

$$\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} y + \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} z = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

How about a line intersecting a subspace:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} t \right\} \text{ intersecting the subspace Span} \left\{ \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \right\}.$$

Example Here's what a nearly fully solved system looks like:

$$x_1 - x_2 + x_3 = 1 \qquad x_2 + 2x_3 = 2 \qquad x_3 = -1,$$

each value written in terms of a constant or in terms of later variables. This makes it easy to roll them up: $x_3 = -1$ so $x_2 = 2 - 2x_3 = 4$ and $x_1 = x_2 - x_3 + 1 = 6$. Done.

Here's a way they can also look:

$$x_1 - x_2 + x_3 = 1 \qquad\qquad x_2 + 2x_3 = 2.$$

You may have stared with more equations, but they got canceled out or something, so you now have 3 variables, 2 equations. In this case, this leads to a free variable, and we'll chose $x_3 = t$.

$$x_2 = 2 - 2t \qquad x_1 = 1 + x_2 - t = 3 - 3t$$

so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 3t \\ 2 - 2t \\ t \end{bmatrix} \quad \Longrightarrow \quad = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} t,$$

which is the parametric form of the solution.

Here's yet another:

$$x_1 - x_2 + x_3 = 1 \qquad\qquad x_2 + 2x_3 = 2 \qquad\qquad 0 = 1$$

a reasonably common reduced problem. It HAS no solution, since it's been broken down into a contradiction $0 = 1$. Whenever you find this one of two things has happened: you've made a mistake, or you're pretty much done, since the problem has no solution.

Here's another:

$$x_1 + 2x_2 - x_3 + 4x_4 = 0 \qquad\qquad x_3 - x_4 = 1$$

which has 4 variables and 2 equations, and NO contradictions. So, we'll set $x_2 = t$, $x_4 = s$ (we have other options, but we'll do it this way). This means $x_3 = 1 + s$. We also get

$$x_1 = -2t + (1 + s) - 4s = 1 - 2t - 3s$$

so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - 2t - 3s \\ t \\ 1 + s \\ s \end{bmatrix} \quad = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix} s.$$

Parametric form of the solution.

The Augmented Matrix

I'm getting tired of typing the $x_1 \ldots x_n$ values. Each system of equations has a fixed shape anyway, why not simplify it? We'll just keep track of the coefficients and the right hand side, keeping all x_k related values in the same column, with rows relating to different equations.

$$\begin{aligned} x_1 - 2x_2 + x_3 - 4x_4 &= 2 \\ x_2 - x_3 + x_4 &= 1 \\ x_3 + x_4 &= 0 \\ x_4 &= 1 \end{aligned}$$

becomes

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

I favor the rightmost one, since it clearly identifies the right hand side constant values as different from the left hand coefficient values.

Notice the pattern, with the equations starting (as in, having non-zero values) further right on the lower rows. This makes it easier to work out solutions. Each row starts with a different variable, x_1 for the first, x_2 for the second, etc. Each row, in turn, includes ONLY the x values used in the lower rows. We start at the bottom, with $x_4 = 1$. Then,

$$x_3 + x_4 = 0 \quad \Longrightarrow \quad x_3 = -x_4 = -1.$$

Next,

$$x_2 = 1 + x_3 - x_4 = 1 - 1 - 1 = -1.$$

Finally,

$$x_1 = 2 + 2x_2 - x_3 + 4x_4 = 2 - 2 + 1 + 4 = 5.$$

Easy enough. If the rows weren't set up like that, we wouldn't be able to do that so easily. As it happens: there's even a name for this sort of arrangement:

Definition: An augmented matrix is in *Row Echelon Form* if the following three requirements are met.

- The first non-zero term in a row is $= 1$.
- All leading ones are to the right of those above them and to the left of those below them.
- Rows of zeros are at the bottom of the matrix.

Definition: Leading ones in a REF augmented matrix are frequently called *Pivots.* A row with a pivot/leading one is called a pivot row. A column with a pivot/leading one is a pivot column and the unknown (x value or whatever) associated with that column is a pivot variable.

Variables that don't have a pivot are typically described as free variables, since the REF augmented matrix is basically written to make them the independent variables. Pivot variables are written as dependent variables.

Examples: The following matrices are in REF:

$$\begin{bmatrix} 1 & 2 & -4 & 12 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Lets write out the general solution of that last one.

First, we have x_4 as a non-pivot variable, so it's a free variable. Make it $x_4 = t$, so $x_3 + x_4 = 2$ so $x_3 = 2 - x_4 = 2 - t$.

Next, x_2 does not have a pivot, so make $x_2 = s$. Our final step is

$$x_1 = 1 + 3x_2 - 2x_3 \implies x_1 = 1 + 3s - 2(2 - t) \implies x_1 = -3 + 3s + 2t.$$

It would have been helpful if the variables with leading ones (the pivot variables) had no other pivot variables in their rows (i.e., they were all zero). This would involve having ONLY free variables accompanying the leading one in a row.

Definition: As augmented matrix is in *Reduced Row Echelon Form* if

- It is in REF.
- Each pivot (leading one) has only zeros above and below (it's the only non-zero term on its column).

Here's one:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

So x_4 is free, make it $x_4 = t$ $x_3 = -t$, $x_2 = 1 - 2t$ and $x_1 = 4 + t$. We got those results earlier, but with more difficulty.

Gaussian Elimination

Named after Gauss, invented in China. That happens.

It's a procedure, an algorithm, which can convert ANY augmented matrix into REF. It's actually quite simple. It uses only three operations applied to the rows.

Row Operations:

1. Row Interchange: take two rows, put one in place of the other and vise versa. Have them switch places.
2. Row Multiplication: take a row and multiply it by a NON-ZERO real number.
3. Row Addition: take a real multiple of a row and add that to ANOTHER ROW. It has to be a different row.

Those three operations are all we need.

Example:

$$\text{Row Interchange:} \quad \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 2 & 0 \end{array}\right] \longrightarrow \quad R_1 \Leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right]$$

$$\text{Row Multiplication:} \quad \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & 2 & 2 \end{array}\right] \longrightarrow \tfrac{1}{2}R_1 \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 2 & 2 \end{array}\right]$$

$$\text{Row Addition:} \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 2 & 2 \end{array}\right] \longrightarrow \quad R_2 - R1 \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 4 & 2 \end{array}\right]$$

Theorem: The three basic row operations each have the following two properties:

- They do not change the solution set to the system.
- They are reversible.

It's easiest to see that with the first operation, row interchange. If you merely list the equations differently, it doesn't change the solution. If you flip two rows, you can reverse by simply flipping them again.

The other two are harder. Take a look, if you like. Hint: there was a relevant question in Assignment 3 to Row Multiplication and Addition not changing the solution.

Definition: two linear systems are said to be *Equivalent* (or, sometimes, *Row Equivalent*) if it is possible to convert one into the other using the basic row operations.

Theorem: two linear systems have the same general solution if and only if they are equivalent.

The Algorithm:

1. Find the leftmost column that contains a non-zero term. If there are none (it's a zero matrix or has no rows at all), you are done.
2. Find the highest row with a non-zero value in that column and use the 'Row Interchange' operation to put that row on the top.
3. Use the 'Row Multiplication' operation to divide by the value there to create a leading one.
4. Use the 'Row Addition' operation on all rows below, making all values directly below the leading one equal to zero.
5. Now start totally ignoring the top row of the system. Go back to step 1, using only the rows below the top one.

Example:

Here's the problem:

$$\left[\begin{array}{rrrr|r} 2 & -2 & -4 & 0 & -4 \\ -1 & 1 & 2 & 1 & 3 \\ 1 & 0 & 0 & -2 & -1 \\ 1 & -2 & -4 & 2 & -3 \end{array}\right].$$

Step 1: the first column has non-zero values. Step 2: we'll take the first row, no need to move anything. Step 3: divide by 2 to give us a leading one on the top left. It is useful to indicate, to the marker and maybe yourself, which operations you are performing. Some examples are given:

$$\tfrac{1}{2}R_1 \qquad (1)/2 \qquad (1)=\tfrac{1}{2}(1) \qquad R_1=\tfrac{1}{2}R_1 \qquad \left[\begin{array}{rrrr|r} 1 & -1 & -2 & 0 & -2 \\ -1 & 1 & 2 & 1 & 3 \\ 1 & 0 & 0 & -2 & -1 \\ 1 & -2 & -4 & 2 & -3 \end{array}\right].$$

Step 4, we will subtract $-1, 1, 1$ times row 1 from rows 2,3 and 4, respectively. Again, It is generally recommended to call your row operations. Two columns with appropriate notation are provided.

$$\begin{array}{ll} & \\ +R_1 & +(1) \\ -R_1 & -(1) \\ -R_1 & -(1) \end{array} \quad \left[\begin{array}{rrrr|r} 1 & -1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & -2 & 1 \\ 0 & -1 & -2 & 2 & -1 \end{array}\right].$$

Notice what we've done here: the leading one on the top row, for the x_1 variable, is now the ONLY x_1 term. We've written x_1 in terms of the others, and now we apply Step 5 and IGNORE that row, and concentrate on the x_1 free system in the lower three rows. The algorithm says 'ignore the row' but we can actually ignore the column as well, since the rows we're looking at from now on have ONLY ZEROS in that column. Here's the system at this point, with a horizontal line indicating what we can ignore:

$$\left[\begin{array}{rrrr|r} 1 & -1 & -2 & 0 & -2 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & -2 & 1 \\ 0 & -1 & -2 & 2 & -1 \end{array}\right].$$

Back to Step 1: the leftmost non-zero column is the second. Step 2: the second row (we're looking at) has a one in that column, so switch them:

$$R_2 \Leftrightarrow R_3 \qquad (2) \Leftrightarrow (3) \qquad \left[\begin{array}{rrrr|r} 1 & -1 & -2 & 0 & -2 \\ \hline 0 & 1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -2 & 2 & -1 \end{array}\right].$$

Step 3: unnecessary, since our leading non-zero term is already a one. Step 4: subtract 0, -1 times the top row (we're paying attention to) from the last two. Leads to

$$\begin{array}{r} \\ \\ \\ +R_2 \end{array} \left[\begin{array}{rrrr|r} 1 & -1 & -2 & 0 & -2 \\ \hline 0 & 1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Step 5 has us ignore that row, which, again, has a leading one that has only zeros under it.

$$\left[\begin{array}{rrrr|r} 1 & -1 & -2 & 0 & -2 \\ 0 & 1 & 2 & -2 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Step 1: the leftmost non-zero term (that we're paying attention to) is on the fourth column. Step 2: it's on the top, no need to switch. Step 3: it's one, no need to divide. Step 4: it has zeros underneath, no need to subtract. Step 5: here's the matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & -2 & 0 & -2 \\ 0 & 1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Step 1: zero matrix, we're done.