Row Reduction, Finished

Recall that we had three row operations

- Row Interchange/Exchange: take two rows and switch their locations.
- Row Multiplication: take a row and multiply it by a *non-zero* real number.
- Row Addition: take two rows, add a real multiple of one to the other. They *have* to be different rows.

Those three operations do not change the solution set of a system of linear equations.

Now for our algorithm to get everything into Row Echelon Form:

1. Find the current matrix's leftmost non-zero column. If there are none, you are done.
2. Find the column's topmost non-zero value. Use Row Interchange to make that row the top row.
3. Use Row Multiplication to make the top row's leading value equal to one.
4. Use Row Addition to make all values directly below that leading one equal to zero.
5. Go back to step 1 using ONLY the rows below the top row.

Example: reduce the following system to row echelon form:

$$\left[\begin{array}{cccc|c} 0 & 1 & -2 & 2 & 3 \\ 3 & 2 & 5 & 1 & 0 \\ 1 & 1 & 1 & 0 & -1 \\ 2 & 1 & 4 & 0 & -1 \end{array}\right]$$

This one illustrates that it can be optimal to not follow the algorithm to the letter. Using the topmost non-zero value in the first column will mean taking that 3, exchanging the top two rows and dividing by 3. That will result in a bunch of annoying fractions on the top row, and require an additional operation that if we simply select the third line instead. There's no reason not too, it's just that the algorithm has to be exact. So, we'll exchange the first and third, getting our leading one, then clear below it:

$$\begin{array}{r} =R3 \\ \\ =R1 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ 3 & 2 & 5 & 1 & 0 \\ 0 & 1 & -2 & 2 & 3 \\ 2 & 1 & 4 & 0 & -1 \end{array}\right] \implies \begin{array}{r} \\ R2-3R1 \\ \\ R4-2R1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ \hline 0 & -1 & 2 & 1 & 3 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & -1 & 2 & 0 & 1 \end{array}\right].$$

Now we ignore the first row, it's got a leading one, now we just need to put the remainder of the matrix in order. So, we'll take the second row and divide by -1 for

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ \hline 0 & 1 & -2 & -1 & -3 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & -1 & 2 & 0 & 1 \end{array}\right] \implies \begin{array}{r} \\ \\ R3=R3-R2 \\ R4=R4+R2 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & -2 & -1 & -9 \\ \hline 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

Now we start ignoring the first TWO rows, and so forth. First: divide the third row by -3 for

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & -2 & 2 & 3 \\ \hline 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \qquad R4 = R4 + 2R3 \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is the actual essence of the method:

1. Arrange for a value of one at the top of the leftmost non-zero column.
2. Clear all values below it to zero.
3. Ignore the row the leading one is in, go back to the first step.

The official algorithm is just more precise. As long as you keep systematic (start at the left, clear below, continue on, etc) you will get to the correct answer.

Categorizing Answers

The REF is not optimal to write out the general solution, but it IS good for figuring out how many solutions you have. Look to where the leading ones (pivots) are:

$$\left[\begin{array}{cccc|c} \mathbf{1} & 1 & 1 & 0 & -1 \\ 0 & \mathbf{1} & -2 & 2 & 3 \\ 0 & 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

These are the key to categorizing the solution. This example has three pivots and four variables. The pivots are at the x_1, x_2 and x_4 columns, meaning we have written it up with x_3 as a free variable and the others as dependent variables. There is no $0 = 1$ type setup, so no contradiction, so there are infinite solutions.

- If there is a pivot on the Right Hand Side (the constant column, with no variables associated with it) then you have a contradiction and no solution.
- If there is no contradiction and any of the variables does not have a pivot then that is a free variable and you have infinitely many solutions.
- If there is no contradiction and every variable has a pivot then every variable is dependent and you get one, unique, solution.

As a result, some examples need only be calculated to the REF to get an answer (ex: Linear Independence questions, spanning questions).

Question: did it have to be x_3 as a free variable in the previous example? The answer is no, in fact. The REF has it as a free variable, but if we had ordered the variables x_1, x_3, x_2, x_4 then we would have gotten x_2 as the free variable. Notice that x_1 and x_2 are both related to x_3's value. However, x_4 IS fixed, no matter which column it's in, it will only have the value 2.

Reduced Row Echelon Form:

This is the upgraded version, remember, with zeros above as well as below the pivots. Our earlier solution had x_1 and x_2 as dependent variables, but x_1 was written in terms of x_2. Here's how you get something into RREF:

1. Use the Gaussian algorithm to get it into REF.
2. Take the rightmost pivot (if there are none, you are done).
3. Use Row Addition to get zeros directly above the pivot.
4. Ignore the pivot's row and anything lower. Go back to step 2 (selecting the row directly above).

So, using our example, we find the rightmost pivot, x_4's, and remove the values above it:

$$R2 = R2 - 2R4 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad R1 - R2 \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Which is, finally, in RREF. Now to get the general solution (the set of all solutions to the system). No contradiction, but there's a free variable $x_3 = t$ (the parameter is not, strictly speaking, necessary, but it's a helpful note of what you're doing). We get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3t \\ -1+2t \\ t \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} t, \qquad t \in \mathbb{R}.$$

And done.

Theorem: a matrix has a single (i.e., unique) row equivalent RREF. As a result, two matrices are row equivalent (meaning it's possible to convert between them using the 3 row operations) if and only if their RREF matrices are equal.

As was stated, some questions are about whether there's a solution (or an infinite number) rather than getting the general solution.

Example: Characterize the span of $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \right\}$.

$$\left[\begin{array}{ccc|c} -1 & 1 & 3 & x \\ 1 & 0 & -1 & y \\ 0 & 2 & 4 & z \end{array}\right]$$

is the system to be solved. First steps:

$$R1 \Leftrightarrow R2 \left[\begin{array}{ccc|c} 1 & 0 & -1 & y \\ -1 & 1 & 3 & x \\ 0 & 2 & 4 & z \end{array}\right] \qquad R2 + R1 \left[\begin{array}{ccc|c} 1 & 0 & -1 & y \\ 0 & 1 & 2 & y+x \\ 0 & 2 & 4 & z \end{array}\right]$$

$$R3 - 2R2 \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & y \\ 0 & 1 & 2 & y+x \\ 0 & 0 & 0 & z-2y-2x \end{array}\right].$$

Notice, this means that the system has a solution if and ONLY if $z - 2y - 2x = 0$. This set of three vectors spans the subspace $\{-2x - 2y + z = 0\}$.

There's an additional thing we can do here. If we have a vector $[x, y, z]$ in the subspace, then there's a solution, etc. Notice, however, that for the arrangement

$$a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

the solution set will be $a = y + c$ and $b = y + x - 2c$, c free (and $z - 2y - 2x = 0$). The value for c is arbitrary. For ANY vector in the span of these vectors, there is a way of expressing it with $c = 0$, so the vector corresponding to c does not contribute to the span. So,

$$\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Example:

Reduce the spanning set for $\mathbb{R}^3$: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$ to a basis.

Answer: You can get rid of any of the last three, if kept in that order you'll get rid of the fourth.

Matrix Operations

Matrix addition and scalar multiplication we have covered (in the $\mathcal{M}_{n\times m}$ space). Recall, however, that addition requires that the matrices be exactly the same size.

Definition: The *Transpose* of a matrix, written A^T, simply reverses the rows and columns. As a result:

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & -2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix}.$$

Definition: Matrix multiplication: it is somewhat complicated. If you have A, an $m \times k$ matrix, and B, a $k \times n$ matrix, then we can multiply them for AB, an $m \times n$ matrix. The values of AB are as follows: the ith row, jth column (element i, j, or $(AB)_{i,j}$) will have a value equal to the dot product of the ith row of A and the jth column of B.

First off, notice this requires that the length of the rows of A must be equal to the length of the columns of B, so the width of A must equal the height of B.

Example: $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} (2)(0)+(-1)(1) & (2)(-2)+(-1)(3) & (2)(-3)+(-1)(0) \\ (0)(0)+(3)(1) & (0)(-2)+(3)(3) & (0)(-3)+(3)(0) \\ (4)(0)+(1)(1) & (4)(-2)+(1)(3) & (4)(-3)+(1)(0) \end{bmatrix} = \begin{bmatrix} -1 & -7 & -6 \\ 3 & 9 & 0 \\ 1 & -5 & -12 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 0+0-12 & 0-6-3 \\ 2+0+0 & -1+9+0 \end{bmatrix} = \begin{bmatrix} -12 & -9 \\ 2 & 8 \end{bmatrix}.$$

Notice the different sizes based on how they were arranged.

Here are two basic matrices:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

and $n \times n$ matrix with all zeros except for ones on the diagonal. Additionally, you have matrices $\mathbf{0}_{n\times m}$ or just $\mathbf{0}$, which are simply the zero vector from $\mathcal{M}_{n\times m}$.

Properties: using matrices A, B, C and real number k. Assume that all multiplications, addition, etc, are defined.

- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$

- $(AB)C = A(BC)$
- $k(AB) = (kA)B = A(kB)$
- $AI = A$ and $IB = B$ if I is the identity matrix
- $\mathbf{0}B = \mathbf{0}$, $A\mathbf{0} = \mathbf{0}$, a matrix times a zero matrix will be another zero matrix (possibly the same size, not always).
- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Finally, we get to some exercises.

Section 1.2: 1.b), 3.b), 4.bdfh)
Section 1.4: 1.bdf), 2.bdfg), 4.