Finishing Eigenvalues

You can also have COMPLEX roots. They can be painful.

Example: Find the eigenvalues/vectors of $\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$.

The characteristic polynomial is $(\lambda - 2)(\lambda) + 2 = \lambda^2 - 2\lambda + 2$. Factoring:

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Now we just need to find the vectors associated with that...

$$\lambda = 1 + i : \qquad \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2}(1+i) & 0 \\ 1-i & 1 & 0 \end{array}\right]$$

eventually breaking down into $\left[\begin{array}{cc|c} 1 & \frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 \end{array}\right]$ for a vector $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$.

$$\lambda = 1 - i : \qquad \left[\begin{array}{cc|c} 1+i & 1 & 0 \\ -2 & -1+i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \frac{1-i}{2} & 0 \\ -2 & -1+i & 0 \end{array}\right]$$

and $\left[\begin{array}{cc|c} 1 & \frac{1-i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$ for a vector $\begin{bmatrix} \frac{1}{2}(1-i) \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$.

Property: Complex eigenvalues from REAL matrices come in conjugate pairs, obviously, but so do their associated eigenvectors.

Confirming this is somewhat tricky, you do it by conjugating each side of $A\mathbf{v} = \lambda\mathbf{v}$ and keep in mind that A is real. You need to use $\overline{zw} = \overline{z}\overline{w}$ and understand that it works for matrix multiplication.

$A\mathbf{v} = \lambda\mathbf{v}$ implies $\overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}}$ which implies $\overline{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$ which simplifies to $A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$. That equation makes $\overline{\mathbf{v}}$ an eigenvector for value $\overline{\lambda}$.

Definition: The subspace $\{\mathbf{x} | A\mathbf{x} = \lambda\mathbf{x}\}$ (it is a subspace) is non-trivial and called an eigenspace if λ is actually an eigenvalue.

So, really, what we're looking for when finding eigenvectors is bases to the eigenspaces. The vectors from different eigenspaces will be linearly independent from each other. The whole set will be L.I. as well if the vectors from individual eigenspaces are kept linearly independent. As in, if you combine valid bases for all the eigenspaces then you will have a linearly independent set. If that set spans $\mathbb{R}^n$ (with A $n \times n$) then you have a potentially helpful property.

Diagonalization

Example: Find the eigenvalues and vectors for $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

$$\begin{aligned} \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & -1-\lambda \end{vmatrix} &= (1-\lambda)(-1-\lambda) - 8 \\ &= \lambda^2 - 1 - 9 \\ &= \lambda^2 - 9 = (\lambda - 3)(\lambda + 3) \end{aligned}$$

so the eigenvalues are ± 3.

$$\lambda_1 = 3 \quad \Longrightarrow \quad \left[\begin{array}{rr|r} -2 & 4 & 0 \\ 2 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or any non-zero multiple.

$$\lambda_2 = -3 \quad \Longrightarrow \quad \left[\begin{array}{rr|r} 4 & 4 & 0 \\ 2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or any non-zero multiple.

Define $P = \left[\begin{array}{c|c} \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right] = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, the (ordered) matrix of the vectors.

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}.$$

$$\begin{aligned} P^{-1}AP &= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 3 & -3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & 0 \\ 0 & -9 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \end{aligned}$$

It's a diagonal matrix composed of the (ordered) eigenvalues. Call this one D.

Definition: The matrix A, $n \times n$, is *diagonalizable* if you can write

$$PDP^{-1} = A \qquad P^{-1}AP = D$$

with D a diagonal matrix.

This definition is equivalent to A having a full set of linearly independent eigenvectors.

Note: being diagonalizable is a totally disconnected property from being invertible.

- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has inverse $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. However, its characteristic polynomial is $\lambda^2 - 2\lambda + 1$, for a double $\lambda = 1$ eigenvalue. Trying to find eigenvectors:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \text{Null}(A - I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

  It is missing a second eigenvector, so it's invertible but not diagonalizable.

- $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible, it has a determinant of zero, etc. It is, however, diagonalizable, with eigenvalues 1 and 0.

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible and already diagonal.

- $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible, and not diagonalizable. It has a double eigenvalue of zero, but only one vector.

Property: A is not diagonalizable if: its characteristic equation has a factor $(\lambda - b)^k$ for eigenvalue b while $(A - bI)$ has fewer than k free variables.

Since the $(A - bI)$ matrix has to have at least one free variable, we get:

Property: if A, $n \times n$, has a full set of n *unique* eigenvalues then A is diagonalizable.

Note that A can be diagonalizable if it has double, triple, etc, root in its characteristic equation. It's just possible it won't be.

A question of this sort involves

- Determining if the matrix is diagonalizable (does it have enough eigenvectors).
- Finding the matrices P, D AND P^{-1}.
- Writing them out in the correct order, so $D = P^{-1}AP$ or $A = PDP^{-1}$.

Doing the actual multiplication is unnecessary, though it is a good check. Also, make sure you've got P, D and P^{-1} in the same order (there is no right order, but they have to be consistent).

Application:

So, is this helpful at all? Well, yes, it can be. Notice:

$$A^2 = \left(PDP^{-1}\right)^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}.$$

We can expand that to $A^k = PD^kP^{-1}$. How does this simplify things? Well:

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & & 0 \\ 0 & 0 & \lambda_3^k & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix},$$

so these are pretty much the easiest matrices to calculate powers.

If we want to find $\lim_{k\to\infty} A^k\mathbf{x}$ then we actually want $P\left(\lim_{k\to\infty} D^k\right)P^{-1}\mathbf{x}$.

Also: those who are continuing into science will probably have to deal with systems of differential equations:

$$\frac{\partial}{\partial t}\mathbf{y} = A\mathbf{y} \qquad \Longrightarrow \qquad \mathbf{y} = e^{At}\,\mathbf{y}(0).$$

Using

$$\begin{aligned} e^{At} = I + At + \frac{A^2}{2}t^2 + \frac{A^3}{6}t^3 + \cdots \quad &= \quad P\left(I + Dt + \frac{D^2}{2}t^2 + \frac{D^3}{6}t^3 \cdots \right)P^{-1} \\ &= P \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & & 0 \\ 0 & 0 & e^{\lambda_3 t} & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} P^{-1}. \end{aligned}$$

Orthogonal Diagonalization

A special case where the eigenvectors are orthogonal. We'll first take a look at a more helpful variant.

Definition: a set of vectors $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_n\}$ is an *orthonormal set* if the set is orthogonal and $|\mathbf{y}_1|^2 = \mathbf{y}_1 \cdot \mathbf{y}_1 = 1$.

It's easy to get these, just take an orthogonal set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and use $\left\{\frac{\mathbf{x}_1}{|\mathbf{x}_1|}, \frac{\mathbf{x}_2}{|\mathbf{x}_2|}, \ldots, \frac{\mathbf{x}_n}{|\mathbf{x}_n|}\right\}$.

Definition: A matrix is orthogonal if it has columns (or rows) composed of an orthonormal set.

Main implication: an orthogonal matrix P will have $P^{-1} = P^T$. For example, look at it in $\mathbb{R}^3$:

$$P^T P = \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \mathbf{y}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

using the orthonormality.

Theorem: The following are equivalent:

- $P^{-1} = P^T$
- The columns of P form an orthonormal basis of $\mathbb{R}^n$
- The rows of P form an orthonormal basis of $\mathbb{R}^n$.

Properties:

- P orthogonal then $\det(P) = \pm 1$.
- P orthogonal then $P^{-1} = P^T$ is orthonormal as well.
- P, Q, both orthonormal then PQ is as well.

One less obvious property: If P is orthonormal then the eigenvalues of P have $|\lambda| = 1$ (they can be complex).

Definition: A is orthogonally diagonalizable if $P^T AP = D$, with D a diagonal matrix and $P^T P = I$.

Example: $A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$.

Notice that $A^T = A$. Not a coincidence.

Theorem:(Principle Axis)

The following are equivalent for A, an $n \times n$ matrix:

- A is orthogonally diagonalizable
- A has a set of eigenvectors that form an orthonormal basis of $\mathbb{R}^n$
- $A^T = A$ (so A is symmetric).

Proving this is rather difficult, but we'll take a look at one bit. If A is orthogonally diagonalizable then:

$$A = PDP^T \quad \Longrightarrow \quad A^T = \left(PDP^T\right)^T = PD^T(P^T)^T = PDP^T = A,$$

so A has $A^T = A$ and is symmetric.

Procedure for orthogonally diagonalizing a matrix is fairly messy, but is much the same as regular diagonalizing. There's one extra element. The eigenvectors for different eigenvalues will be orthogonal, but those from the same eigenvalue will probably be only linearly independent. When you find an eigenvalue with multiple roots in the characteristic equation, you'll have to use Gram-Schmidt on the resulting vectors.

Example: Diagonalize $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (hint: 1 is an eigenvector).

Exercises:

Section 2.3: 2.bdf), 6.b), 7.bdfh), 8, 17.b)

Section 4.7: 2.bdf), 5.bc) (c is messy)