Determinants, Finished

We ended with $\det(AB) = \det(A)\det(B)$. We actually can prove a little more using this.

The effect on $\det(A)$ of a Row Operation on A

- Row Interchange on A converts $\det(A)$ to $-\det(A)$.
- Row Multiplication by $b \in \mathbb{R}$ converts $\det(A)$ to $b\det(A)$.
- Row Addition does nothing to the determinant.

Note: this can be proven using the $\det(AB) = \det(A)\det(B)$ equality. You only need to construct a matrix B such that BA would be equal to A with the row operation applied. For instance, Row Interchange of R_1 and R_2 on a 3×3 matrix A is equivalent to

$$BA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

Note that $\det(B) = -1$.

Implication 1: We can use row operations to simplify the determinant.

Example:

$$\begin{vmatrix} 5 & -1 & 8 & -4 \\ 2 & 1 & -2 & 1 \\ 3 & -1 & 4 & -1 \\ -1 & 1 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 0 & -1 \\ 2 & 1 & -2 & 1 \\ 3 & -1 & 4 & -1 \\ 5 & -1 & 8 & -4 \end{vmatrix}$$
$$= -\begin{vmatrix} -1 & 1 & 0 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & 2 & 4 & -4 \\ 0 & 4 & 8 & -9 \end{vmatrix}$$
$$= 2\begin{vmatrix} -1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 3 & -2 & 1 \\ 0 & 4 & 8 & -9 \end{vmatrix}$$
$$= 2\begin{vmatrix} -1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 0 & -1 \end{vmatrix}.$$

So, we have a triangular matrix here (it has only zeros below the main diagonal).

Property: The determinant of a triangular matrix (either upper or lower triangular) is just the product of the diagonal.

So:

$$2\begin{vmatrix} -1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 2(-1)(1)(-8)(-1) = -16$$

Things to note:

$$\det(bA) = a^n \det(B), \qquad a \in \mathbb{R}, \quad B \text{ is } n \times n.$$

$$\det(A^T) = \det(A).$$

$$\det(A) + \det(B) \neq \det(A + B), \qquad \text{at least not very often.}$$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ has } \det(A+B) = 0 \qquad \det(A) + \det(B) = 0.$$

Implication 2: none of the officially recognized Row Operations will multiply the determinant by zero. As a result: if A, B are Row Equivalent then

$$\det(A) = 0 \Longleftrightarrow \det(B) = 0 \quad , \quad \det(A) \neq 0 \Longleftrightarrow \det(B) \neq 0.$$

Here's something else: $\det(I) = 1$. So:

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

Property: $\det(A^{-1}) = \frac{1}{\det(A)}$.

This implies:

$$\det(A) = 0 \quad \Longrightarrow \quad A \text{ not invertible.}$$

Furthermore, if A is not invertible then (using the $\left[\begin{array}{c|c} A & I \end{array}\right]$ algorithm) does not reduce to I on the left. As a result, it reduces to something with a row or column of zeros, so determinant zero. As a result:

$$A \text{ not invertible} \quad \Longrightarrow \quad \det(A) = 0$$

Theorem:

$$A \text{ not invertible} \quad \Longleftrightarrow \quad \det(A) = 0.$$

$$A \text{ invertible} \quad \Longleftrightarrow \quad \det(A) \neq 0.$$

We can also say that $\det(A) = 0$ if and only if A has a missing pivot (as in, not n of them) etc.

Eigenvalues/Eigenvectors

Definition: the vector $\mathbf{v} \neq \mathbf{0}$ is an eigenvector for A, and $\lambda \in \mathbb{R}$ its associated eigenvalue, if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

The matrix A simply RE-SCALES (re-sizes, etc) the vector $\mathbf{v}$.

Example: $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ has eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvector $\lambda_1 = 0$:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with eigenvalue $\lambda_2 = 1$ with

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Example: $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ -2 & 0 & 1 \end{bmatrix}$ has $\lambda_1 = -1$ for $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 5 \\ -2 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} -2 \\ 5 \\ -2 \end{bmatrix}.$$

Thing to note: if $\mathbf{v}$ and λ are an eigenvector/eigenvalue pair then for any $b \in \mathbb{R}$ we get

$$A(b\mathbf{v}) = bA\mathbf{v} = b\lambda\mathbf{v} = \lambda(b\mathbf{v}).$$

So, ANY REAL MULTIPLE of $\mathbf{v}$ will also be an eigenvector. As a result, we are interested in *directions* more than anything else, so linear independence is an issue.

How To Find Them:

If λ is an eigenvalue then we have $\mathbf{v} \neq \mathbf{0}$ so that

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Longrightarrow \quad A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}.$$

This means

$$A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad (A - \lambda I)\mathbf{v} = \mathbf{0}$$

which means λ is an eigenvalue if and only if $(A - \lambda I)$ has a non-trivial Null space.

$$\text{Null}(A - \lambda I) \quad \text{non-trivial} \quad \Longrightarrow \quad (A - \lambda I) \quad \text{not-invertible.}$$

One more step leads to

$$\lambda \text{ an eigenvalue of } A \quad \Longleftrightarrow \quad \det(A - \lambda I) = 0.$$

Note that, as usual, the determinant outputs a number. However, with a variable (λ) in it, the determinant becomes a polynomial.

Definition: The polynomial $\det(A - \lambda I)$ is the CHARACTERISTIC POLYNOMIAL of A.

- The value of the characteristic polynomial of A with $\lambda = 0$ is the determinant of A.
- The FACTORS of the char. poly. are the eigenvalues of A.

So, our procedure for finding eigenvectors/values of A is

1. Calculate the Characteristic Polynomial of A ($\det(A - \lambda I)$)
2. Find the eigenvalues, the roots of the char. poly.
3. Find bases for the null spaces of $A - \lambda I$, with λ the eigenvalues.

Property: If A is a triangular matrix (upper or lower) then its eigenvalues are simply its diagonal terms.

Example: $A = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 9 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$ has eigenvalues 1,9,0 and 12.

Now for a better example...

Example: Find the eigenvalues of $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ -2 & 0 & 1 \end{bmatrix}$ (recall that we already have -1).

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -2 \\ 2 & 1-\lambda & 3 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 2 & 1-\lambda & 3 \\ -2 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)\left((1-\lambda)^2 - 4\right)$$
$$= (1-\lambda)\left(\lambda^2 - 2\lambda - 3\right).$$

This leads to $(1-\lambda)(\lambda-3)(\lambda+1)$ with factors $1, -1, 3$. So, those are the eigenvalues.

To find the eigenvectors, simply check the null spaces of $A - \lambda I$, with λ an eigenvalue. Since we need L.I. vectors, this is *exactly the same* as finding basis vectors for the null space.

$$\text{For } \lambda_1 = 1: \qquad \text{solve} \left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 2 & 0 & 3 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The vector is any multiple (but zero) of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$$\text{For } \lambda_2 = 3: \qquad \text{solve} \left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 2 & -2 & 3 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so the vector is any (non-zero) multiple of $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

$$\text{For } \lambda_3 = -1 \text{ we had } \begin{bmatrix} -2 \\ 5 \\ -2 \end{bmatrix}.$$

Example Questions:

Section 2.1: 6.b), 10.bdf), 12.bd)

The eigenvalue section in the text mostly covers the next subject, diagonalization. If you want eigenvector practice, take a look at 2.bdf), but only find the eigenvalues/vectors.