Non-Properties: (things we'd expect to be true, but aren't).

- $AB \neq BA$ sometimes.
- $A^2 = \mathbf{0}$, with A not equal to a zero matrix.
- $AB = 0$, with neither a zero matrix.
- $AC = BC$ even though $A \neq B$ and C not a zero matrix.

So, matrix multiplication is not as reliable as real number multiplication. This is to be expected considering the complexity of the elements. So, why use that definition? Well, here's one good reason:

$$\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & 1 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

is equivalent to the problem

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -3x_1 + x_2 + x_3 &= 2 \\ 2x_2 - 2x_3 &= -1 \\ x_1 - x_3 &= 0. \end{aligned}$$

or

$$\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \\ -2 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}.$$

Property: Any linear system can be written in the form $A\mathbf{x} = \mathbf{b}$, with A a real matrix.

Matrix Derived Spaces

These are spaces, all subspaces/vector spaces, derived from a matrix. We'll call it A, it'll be $n \times m$. We will look into three, how to find bases for them, etc. The first two are the most difficult to work with, but the procedures will be very familiar. The last is the easiest, but will take some explanation.

Definition: The *Column Space* of a matrix A, $\text{Col}(A)$, is the span of the columns of A.

$\text{Col}(A)$ is a span, so it's a subspace. You can write it two ways:

$$\text{Col}(A) = \text{Span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n\} = \{A\mathbf{x}, \text{ for all } \mathbf{x} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n.$$

(with $A = [\mathbf{a}_1|\mathbf{a}_2|\mathbf{a}_3|\cdots|\mathbf{a}_n]$). So, you can view it as a span, or as the 'output' of the operator $\mathbf{x} \to A\mathbf{x}$.

How to find the basis for Col(A): this is easy. We discussed this earlier. You row reduce A down to REF, and the pivot columns relate to the basis vectors. You have to use the ORIGINAL columns, but the row reduction decides which are necessary to the span (so, the minimal spanning set).

Theorem: The columns of A that hold pivots form a basis of Col(A).

Proof:

Fairly simple, and mostly involves reiterating stuff we've already seen. Take the linear system $[A|\mathbf{y}]$ and solve it. Removing the columns without pivots will not affect the span, since for any $\mathbf{y}$ where you have a solution you'd also have a solution with all the free variables set to zero. So, cut A back to B, which only has the pivot columns. Solving $[B|\mathbf{0}]$ will result only in pivot columns on the left, so the trivial solution, and the columns of B form a linearly independent spanning set and so a basis.

THE thing to remember is that the basis is the ORIGINAL column vectors, those in A as it started. The row reduction changes their shape.

Corollary: The dimension of Col(A) is equal to the number of pivots in A.

Example:

$$\text{Find a basis for } \operatorname{Col}(A) \text{ with } A = \begin{bmatrix} -1 & -1 & 0 & 2 & 2 \\ 2 & 1 & -1 & -1 & -1 \\ 1 & 0 & -1 & 2 & 3 \end{bmatrix}.$$

This one reduces to

$$\begin{bmatrix} 1 & 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \text{ so we have the basis } \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

And now for another space. This one is really only definable using the 'A as operator' setup.

Definition: The *Null Space* of A is the set

$$\operatorname{Null}(A) = \{\mathbf{x} \text{ such that } A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^m.$$

This is just a specific version of the Linear Independence equation for the columns of A. The order of the vectors is essential, however...

Finding the basis for this is not that easy, but it should be familiar. We just want the parametric vector form of the solution of $A\mathbf{x} = \mathbf{0}$. We've done this before.

Example: Find a basis for Null(A) using the A from before.

We can start with it in REF, but we are best to get it into RREF to do this part. It further reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{so the Null space is } = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} t+s \\ -t-3s \\ t \\ -2s \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} s$$

$$\text{so the basis is } \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

This one is always writable as a span, and so is a subspace. You can prove it using the subspace test if you like.

One more thing to notice: our example here has dimensions 3×5. Its Column Space has dimension 3 and its Null space has dimension 2. These sum up to its width $= 5$. This is to be expected: the dimension of the column space is the number of pivots, the dimension of the null space is the number of free variables, which is the pivot-less rows.

Property: the dimension of the Null space of A is the number of pivot-less rows. As such, the dimensions of the column and null spaces sum up to the number of columns in A.

Definition: The *Row Space* of a matrix A is the span of the rows of A.

This one is somewhat different from the others. Here's one good reason:
Property: Row operations on a matrix do not change the Row Space.

This is fairly easy to see. Row operations are a restricted form of vector operations, and so a span can undo any row operation. Also, if a given row gets removed during row reduction (i.e., made into a zero row) that means it's a linear combination of the remaining non-zero rows, and so does not contribute to the span. As a result, our basis will be the surviving rows after row reduction. They are a spanning set (being equal in span to the originals) and are quite trivially linearly independent.

Property: The basis for Row(A) is the pivot rows from the REF (or RREF) of A.

Notice that we use the rows directly from the reduced form. NOT THE ORIGINAL ROWS. The interchange operation can move them around...

Example:

$$\text{Find a basis to the row space of } A = \begin{bmatrix} -2 & -1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & -2 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

This reduces to

$$\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which makes the basis

$$\{[1, 0, -2, -3], [0, 1, 2, 4], [0, 0, 1, 1]\}\,.$$

Definition: The *Rank* of A, with A an $n \times m$ matrix, is:

$$\begin{aligned} \text{Rank}(A) &= \text{ number of pivots of } A \\ &= \text{Dim}(\text{Col}(A)) \\ &= \text{Dim}(\text{Row}(A)) \\ &= m - \text{Dim}(\text{Null}(A)). \end{aligned}$$

Relation to Solution Sets:

The matrix size, rank, etc, can be used to determine the number of solutions, and even the existence of solutions. First, if a matrix A, $n \times m$, has $\text{Rank}(A) = n$ then no problem $[A|\mathbf{b}]$ will have a pivot on the right hand side. As a result, there will be a solution. At least *a* solution. If $\text{Null}(A)$ has dimension > 0, if $\text{Rank}(A) < m$, then you get infinite solutions:

Property: if $\mathbf{x}$ is a solution to $[A|\mathbf{b}]$ and $\mathbf{y} \in \text{Null}(A)$ then $\mathbf{x} + \mathbf{y}$ is also a solution of $[A|\mathbf{b}]$.

Recall that if $\text{Null}(A)$ has zero dimension (no free variables) then $\text{Null}(A)$ has precisely ONE element ($\mathbf{0}$). If $[A|\mathbf{b}]$ has a unique solution, A must have as many pivots as columns.

Exercises

Section 1.2: 9.bd)

Section 1.3: 1.bdfh) Note: Coefficient matrix means Left Hand Side, so the coefficient matrix is the matrix given. The Right Hand Side will be all zeros, in this case, 4.bd), 6, 7.bdf)

Section 4.4: 1.bd), 3.b), 5.b), 7.bdf), 12, 16.b)