Solution to Final Examination
MAT1320, Fall 2011

I. Multiple-choice Questions (2 × 7 = 14 marks)

1. Which one of the following values is closest to the definite integral $\int_{-\pi/3}^{\pi/4} \sin^3\theta\cos\theta d\theta$?

(A) 0; (B) –0.078; (C) 1; (D) 1.03; (E) –1.5; (F) 0.061.

Solution. (B) Use variable substitution $u = \sin\theta, u' = \cos\theta$. $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

Then

$$\int_{-\pi/3}^{\pi/4} \sin^3\theta\cos\theta d\theta = \int_{-\sqrt{3}/2}^{\sqrt{2}/2} u^3 du = \frac{1}{4}\left[u^4\right]_{u=-\sqrt{3}/2}^{\sqrt{2}/2} = \frac{1}{4}\left(\frac{4}{16} - \frac{9}{16}\right) = -\frac{5}{64} \approx -0.078.$$

Note that this question can also be solved by substitution $u = \cos\theta$ and the formula $\sin^2\theta = 1 - \cos^2\theta$.

2. Consider the function $f(x) = x^4 + 4x^3 + 6x^2 + 24x + 24$. Which one of the following statements is true?

(A) $f(x)$ attains a local maximum at $x = -1$.
(B) $f(x)$ attains a local minimum at $x = -1$.
(C) $x = -1$ is a critical number of $f(x)$, but $f(x)$ does not have a local maximum or a local minimum at $x = -1$.
(D) $x = -1$ is not a critical number of $f(x)$, but the graph of $f(x)$ has an inflection point at $x = -1$.
(E) $x = -1$ is not a critical number of $f(x)$, and the graph of $f(x)$ does not have an inflection point at $x = -1$.

Solution. (E) $f'(x) = 4x^3 + 12x^2 + 12x + 24$. Since $f'(-1) = -4 + 12 - 12 + 24 \neq 0$, $x = -1$ is not a critical number of $f(x)$.

$f''(x) = 12x^2 + 24x + 12 = 12(x + 1)^2$. Let $f''(x) = 0$. $x = -1$. When $x < -1$ or $x > 1$, $f''(x) > 0$. The graph of $f(x)$ is concave up on both side of $x = -1$, $f(x)$ does not have an inflection point at $x = -1$.

3. Suppose $y = h(x)$, where $h(x) = f(g(x))$, is a composite function such that $g(2) = 3$, $g'(2) = 4$, $f(2) = 3$, $f'(2) = 6$, and $f'(3) = 5$. Find the value of $\frac{dy}{dx}$ at $x = 2$.

(A) 20; (B) 24; (C) 12; (D) 30; (E) 18; (F) 15.

Solution. (A) Let $u = g(x)$. Then $y = f(u)$. By the chain rule,

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. When $x = 2$, $u = g(2) = 3$, $\frac{du}{dx} = g'(2) = 4$, $\frac{dy}{du} = f'(3) = 5$. $\frac{dy}{dx} = 20$.

4. Evaluate definite integral $\int_0^1 \frac{1}{x^2+1}dx$.

(A) $\ln 2$; (B) π; (C) 1; (D) $\pi/4$; (E) $1/2$; (F) $\pi/2$.

Solution. (D) $\int_0^1 \frac{1}{x^2+1}dx = [\arctan x]_{x=0}^1 = \frac{\pi}{4}$.

5. On which interval(s) is the graph of the function $f(x) = (x^2 - 7x + 14)e^x$ concave up?

(A) It never is; (B) $(2, \infty)$ only; (C) $(-\infty, 1)$ and $(2, \infty)$ only;
(D) It always is; (E) $(1, 2)$ only; (F) $(-\infty, 1)$ only.

Solution. (C) $f'(x) = (2x - 7)e^x + (x^2 - 7x + 14)e^x = (x^2 - 5x + 7)e^x$.
$f''(x) = (2x - 5)e^x + (x^2 - 5x + 7)e^x = (x^2 - 3x + 2)e^x$.

Since $x^2 - 3x + 2 > 0$ for $x < 1$ or $x > 2$, the graph of this function is concave up in $(-\infty, 1)$ and $(2, \infty)$.

6. If $y = f(x)$ is a continuous function such that $\int_1^7 f(x)dx = 9$, and $\int_4^7 f(x)dx = 15$, what is $\int_1^4 (2f(x) - 5)dx$?

(A) -33; (B) -27; (C) -22; (D) 0; (E) 7; (F) 21.

Solution. (B) $\int_1^4 (2f(x)-5)dx = 2\int_1^4 f(x)dx - 5\int_1^4 dx = 2\left(\int_1^7 f(x)dx - \int_4^7 f(x)dx\right) - 15 = -27$.

7. The equation of the tangent line of the graph of $x^2y + 2xy - y^2 - 3x + 1 = 0$ at the point where $x = 1$ and $y = 2$ is

(A) $y = 3x - 1$; (B) $y = 5x - 3$; (C) $y = -\frac{1}{2}x + \frac{5}{2}$;

(D) $y = -2x + 4$; (E) $y = x + 1$; (F) $y = \frac{3}{2}x + \frac{1}{2}$

Solution. (B) $2xy + x^2y' + 2y + 2xy' - 2yy' - 3 = 0$. When $x = 1$ and $y = 2$, $5 - y' = 0$. $y' = 5$.
The equation of the tangent line is $y = 5(x - 1) + 2$, or $y = 5x - 3$.

8. Consider a function $y = f(x)$. Which one of the following statement is true?

(A) If $f'(a) = 0$ or $f'(x)$ is not defined at $x = a$, then $x = a$ is a critical number of $f(x)$.
(B) If $f(x)$ attains an absolute maximum at $x = a$, then $x = a$ is a critical number of $f(x)$.
(C) If $x = a$ is a critical number of $f(x)$, then $f(x)$ attains a local maximum or a local minimum at $x = a$.
(D) If $f(x)$ attains a local maximum or a local minimum at $x = a$, then $x = a$ is a critical number of $f(x)$.
(E) If $(a, f(a))$ is an inflection point of the graph of $f(x)$, then $f''(a) = 0$.
(F) If function $f''(a) = 0$, then $(a, f(a))$ is an inflection point of the graph of $f(x)$.

Answer. (D).

(A) is false because, if $x = a$ is not in the domain of $f(x)$, it is not a critical number even if $f'(a) = 0$ or $f'(a)$ does not exist.
(B) is false because an absolute maximum or an absolute minimum may be attained at an end of the domain of $f(x)$.
(C) is false because, if $f'(x)$ has the same sign on both sides of $x = a$, then $f(x)$ does not attain a local maximum or a local minimum at $x = a$.
(E) is false because $f''(x)$ may not be defined at an inflection point.
(F) is false because, if $f''(x)$ has the same sign on both sides of $x = a$, then the graph of $f(x)$ does not have an inflection point at $x = a$.

9. Let $y = (\sqrt{x})^{\sqrt{x}}$. What is $f'(4)$?

(A) $\ln 2 + 1$; (B) $4 \ln 2$; (C) 2; (D) 0; (E) 3 / 2; (F) 4.

Solution. (A) $\ln y = \sqrt{x}\ln\sqrt{x}$. $y'/y = \frac{1}{2\sqrt{x}}\ln\sqrt{x} + \frac{1}{2x}\sqrt{x} = \frac{\ln\sqrt{x}+1}{2\sqrt{x}}$.

$$y' = (\sqrt{x})^{\sqrt{x}}\left(\frac{\ln\sqrt{x}+1}{2\sqrt{x}}\right) = \frac{1}{2}\left(\sqrt{x}\right)^{\sqrt{x}-1}(\ln\sqrt{x}+1).$$

When $x = 4$, $y' = \frac{1}{2}(\sqrt{4})^{\sqrt{4}-1}(\ln\sqrt{4}+1) = \ln 2 + 1$.

10. A farmer has 84 meters of fence, and he wants to use the fence to enclose a rectangular land. With different dimensions, this rectangular land may have different area. What is the largest rectangular area that can be enclosed by 84 meters of fence?

(A) 7056 m^2; (B) 1764 m^2; (C) 84 m^2;
(D) 441 m^2; (E) 4096 m^2; (F) 517 m^2.

Solution. (D) Let the rectangular area has width x and depth y. Then the area is $A = xy$. Since $2x + 2y = 84$. $x + y = 42$. $y = 42 - x$. $A = x(42 - x)$. $A' = 42 - 2x$. Let $A' = 0$. $x = 21$. Since

$A' > 0$ when $x < 21$, and $A' > 0$ when $x > 21$. Area A attains a local maximum at $x = 21$, which is also an absolute maximum. The maximum value of $A = 21 \times (42 - 21) = 441$ m^2.

Part II. Short Answer Questions (2 × 7 = 14 marks)

1. $F(x) = \int_{x^2}^{1} t \arcsin t \, dt$ is a function of x. Find the derivative of this function.

Solution. $\int_{x^2}^{1} t \arcsin t \, dt = -\int_{1}^{x^2} t \arcsin t \, dt$. Let $u = x^2$. By the fundamental Theorem of Calculus and the chain rule, $F'(x) = \left(\frac{d}{du}\left(-\int_{1}^{u} t \arcsin t \, dt\right)\right)\left(\frac{du}{dt}\right) = -(u \arcsin u)(2x) = -2x^3 \arcsin (x^2)$.

2. Suppose Newton's method is being used to find a root of the equation $x^3 - 3x + 1 = 0$ with an initial approximation $x_0 = 1.5$. What is the value of the next approximation x_1? (Use at least 4 digits after the decimal point in your calculation).

Solution. $x_1 = x_0 - \frac{x_0^{\ 3} - 3x_0 + 1}{3x_0^{\ 2} - 3} \approx 1.5333$.

3. What values in the interval $[0, 2\pi]$ are the critical numbers of the function $y = \sqrt{3}\, x + 2 \cos x$?

Solution. $y' = \sqrt{3} - 2\sin x$. Let $y' = 0$. $\sin x = \sqrt{3} / 2$. In interval $[0, 2\pi]$, $x = \pi / 3$ and $2\pi / 3$.

4. Using the velocity data in the table below, compute an overestimate and an underestimate of the distance the car travels in 10 second from $t = 0$ to $t = 10$ by the left and the right sums. Include the unit of measurement in your answer.

t (seconds)	0	2	4	6	8	10
v (m / s)	32	24	20	17	11	7

Solution. The left sum is 2 × (32 + 24 + 20 + 17 + 11) = 208 m.
The right sum is 2 × (24 + 20 + 17 + 11 + 7) = 158 m.

5. Find the linear approximation of the function $f(x) = \frac{1}{\sqrt{x+2}}$ near $x = 7$, and use this result to estimate $\frac{1}{\sqrt{10}}$. Provide your answer to 4 decimal places.

Solution. $f(x) = (x + 2)^{-1/2}$. $f'(x) = -\frac{1}{2}(x+2)^{-3/2}$. When $x = 7$, $f'(7) = -\frac{1}{54}$. The linear approximation has the form $y = -\frac{1}{54}x + b$. When $x = 7$, $y = \frac{1}{3}$. Hence, $\frac{1}{3} = -\frac{7}{54} + b$, $b = \frac{25}{54}$.

The linear approximation is $y = -\frac{1}{54}x + \frac{25}{54}$. To estimate $\frac{1}{\sqrt{10}} = \frac{1}{\sqrt{8+2}}$, $x = 8$.

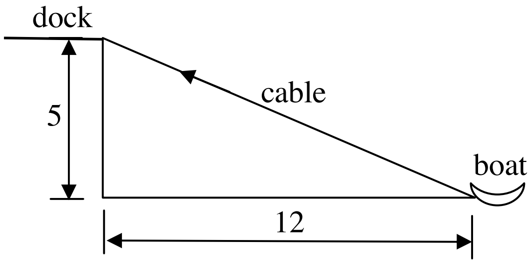
$\frac{1}{\sqrt{10}} \approx -\frac{8}{54} + \frac{25}{54} = \frac{17}{54} \approx 0.3148$.

6. Use L'Hospital's Rule to show that the limit $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \frac{1}{2}$.

Solution.

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1}\frac{\ln x + 1 - 1}{\ln x + \frac{x-1}{x}} = \lim_{x\to 1}\frac{x\ln x}{x\ln x + x - 1} = \lim_{x\to 1}\frac{\ln x + 1}{\ln x + 2} = \frac{1}{2}.$$

7. The surface of a dock is 5 meters above the surface of water. A boat on the water is pulled in by a cable towards the dock. When the boat is 12 meters away horizontally from the dock, it is approaching the dock horizontally at a rate of 0.5 meters per second. How fast is the cable being pulled in?



Solution. Let the horizontal distance between the dock and the boat be x, and let the length of the cable be y. Both x and y are functions of time t. Then $y^2 = x^2 + 5^2$. Take the derivative with respect to t: $2yy' = 2xx'$. Hence, $y' = xx' / y$. When $x = 12$, $y = 13$.
Since $x' = -0.5$, $y' = -0.5 \times 12 / 13 = -6 / 13 \approx -0.46$ m/s.

Part III. Long Answer Questions (16 marks)

1. (3 marks) The surface area and the volume of a circular cylinder with radius r and height h are $A = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$, respectively. If the surface area is fixed to be $2400\pi\,\text{cm}^2$, find the radius r and height h that maximizes the volume.

Solution. Since $2\pi r^2 + 2\pi rh = 2400\pi$, $h = (1200\pi - \pi r^2) / (\pi r) = (1200 - r^2) / r$.
$V = \pi r^2 h = 1200\pi r - \pi r^3$, $0 < r < \infty$. $V' = 1200\pi - 3\pi r^2$. Let $V' = 0$.

$r = \sqrt{\dfrac{1200\pi}{3\pi}} = \sqrt{400} \approx 20$ cm. $h = (1200 - 400) / 20 = 40$ cm. Since $V' > 0$ when $0 < r < 20$, and $V' < 0$ when $20 < r$, the volume V is increasing from 0 to 20 and F is decreasing from 20 to infinity. Therefore, V attains an absolute maximum at $r = 20$. The maximum volume is $V = \pi r^2 h = 16000\pi\ \text{cm}^3$.

2. (10 marks) Find the following definite or indefinite integrals:

(a) $\displaystyle\int_1^{e^3} \frac{1}{x\sqrt{1+\ln x}}\,dx$.

Solution. Let $u = \ln x + 1$. Then $u' = 1 / x$. When $x = 1$, $u = 1$, and when $x = e^3$, $u = 4$.

$$\int_1^{e^3} \frac{1}{x\sqrt{1+\ln x}}\,dx = \int_1^4 \frac{1}{x\sqrt{1+\ln x}}(x)du = \int_1^4 \frac{1}{\sqrt{u}}du = 2\Big[\sqrt{u}\Big]_{u=1}^4 = 2.$$

(b) $\displaystyle\int_1^e x\ln x dx$.

Solution. Use integration by parts.

$$\int_1^e x\ln x dx = \int_1^e \ln x d\left(\frac{x^2}{2}\right) = \left[\frac{x^2}{2}\ln x\right]_{x=1}^e - \int_1^e \frac{x^2}{2}\left(\frac{1}{x}\right)dx = \frac{e^2}{2} - \frac{1}{2}\int_1^e x dx = \frac{e^2}{2} - \frac{1}{4}(e^2 - 1) = \frac{1}{4}(e^2 + 1).$$

(c) $\displaystyle\int \frac{5x+13}{x^2-2x-3}dx$.

Solution. Use partial fraction. $\dfrac{5x+13}{x^2-2x-3} = \dfrac{A}{x-3} + \dfrac{B}{x+1} = \dfrac{A(x+1)+B(x-3)}{(x-3)(x+1)}$.

Then $A(x + 1) + B(x - 3) = 5x + 13$. Let $x = -1$. $-4B = 8$, $B = -2$. Let $x = 3$. $4A = 28$, $A = 7$.

$$\int \frac{5x+13}{x^2-2x-3}dx = 7\int \frac{1}{x-3}dx - 2\int \frac{1}{x+1}dx = 7\ln|x-3| - 2\ln|x+1| + C.$$

(d) $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}dx$.

Solution. Let $x = 2\sin u$, $-\pi/2 \le u \le \pi/2$. Then $x' = 2\cos u$. $\sqrt{4-x^2} = 2\cos u$.

$$\int \frac{1}{x^2\sqrt{4-x^2}}dx = \frac{1}{8}\int \frac{1}{\sin^2 u\cos u}(2\cos u)du = \frac{1}{4}\int \frac{1}{\sin^2 u}du = -\frac{1}{4}\cot u + C = -\frac{\sqrt{4-x^2}}{4x} + C.$$

(e) $\int \frac{e^{x^{1/3}}}{x^{2/3}}\,dx$.

Solution. Let $u = x^{1/3}$. Then $u' = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$.

$$\int \frac{e^{x^{1/3}}}{x^{2/3}}\,dx = \int \frac{e^{x^{1/3}}}{x^{2/3}}\left(3x^{2/3}\right)du = 3\int e^{u}\,du = 3e^{u} + C = 3e^{x^{1/3}} + C.$$

3. (3 marks) Approximate the definite integral $\int_1^4 \frac{1}{x}\,dx$ to 4 decimal places using the Trapezoidal rule with $n = 6$ subdivisions. What is the true value?

Solution. $h = 0.5$.

x	$1/x$
1.0	1.0000
1.5	0.6667
2.0	0.5000
2.5	0.4000
3.0	0.3333
3.5	0.2857
4.0	0.2500

$$T(6) = \frac{0.5}{2}(1 + 2 \times 0.6667 + 2 \times 0.5 + 2 \times 0.4 + 2 \times 0.3333 + 2 \times 0.2857 + 0.25) = 1.4053.$$

The true value is $\int_1^4 \frac{1}{x}\,dx = [\ln x]_{x=1}^{4} = \ln 4 \approx 1.3863$.