University of Ottawa
ECO 3145 Mathematical Economics I
Fall 2012, Professor Shiell

Exam I – Answers

1. $L + 2W = 32 \quad \Rightarrow \quad L = 32 - 2W$

   and

   $A = LW$

   Substitution:

   $$\begin{aligned} A &= (32 - 2W)W \\ &= 32W - 2W^2 \end{aligned}$$

   Choose W to maximize A. This requires finding the stationary point.

   $$\frac{\partial A}{\partial W} = 32 - 4W = 0 \quad \Rightarrow \quad W = 8$$

   Substitution:

   $L = 32 - 2W = 32 - 2(8) = 16$.

   So the stationary point is $(W = 8, L = 16)$.

   To check whether it is a local maximum or minimum, check the second derivative at the stationary point.

   $$\frac{\partial^2 A}{\partial W^2} = -4 < 0$$

   Since the second derivative is negative, the stationary point is a local maximum.

   (Actually, since the derivative has the same sign regardless of the point at which it is evaluated, we can conclude that the stationary point is a global maximum.)

2. Consider the function $f(x_1, x_2) = 100 - 5x_1 + 4x_1^2 - 9x_2 + 5x_2^2 + 8x_1x_2$.

a.) Derive the first-order conditions.

$$f_1 = -5 + 8x_1 + 8x_2 = 0$$

$$f_2 = -9 + 10x_2 + 8x_1 = 0$$

Rewriting:

$$8x_1 + 8x_2 = 5$$
$$8x_1 + 10x_2 = 9$$

In matrix form:

$$\begin{bmatrix} 8 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Call the coefficient matrix A, i.e. $A = \begin{bmatrix} 8 & 8 \\ 8 & 10 \end{bmatrix}$.

Now solve for x_1 and x_2 using Cramer's rule (alternatively you can compute the inverse matrix A^{-1} and solve the system for the x vector).

Cramer's rule:

$$x_1 = \frac{\begin{vmatrix} 5 & 8 \\ 9 & 10 \end{vmatrix}}{|A|} \qquad x_2 = \frac{\begin{vmatrix} 8 & 5 \\ 8 & 9 \end{vmatrix}}{|A|}$$

Note $|A| = 80 - 64 = 16$

Therefore:

$$x_1 = \frac{50 - 72}{16} = -\frac{11}{8} \qquad x_2 = \frac{72 - 40}{16} = \frac{32}{16} = 2$$

b.) Construct the Hessian matrix:

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}.$$

$f_{11} = 8$ $\quad$ $f_{12} = 8$ $\quad$ $f_{21} = 8$ $\quad$ $f_{22} = 10$

$$H = \begin{bmatrix} 8 & 8 \\ 8 & 10 \end{bmatrix}$$

$LPM_1 = |8| = 8$

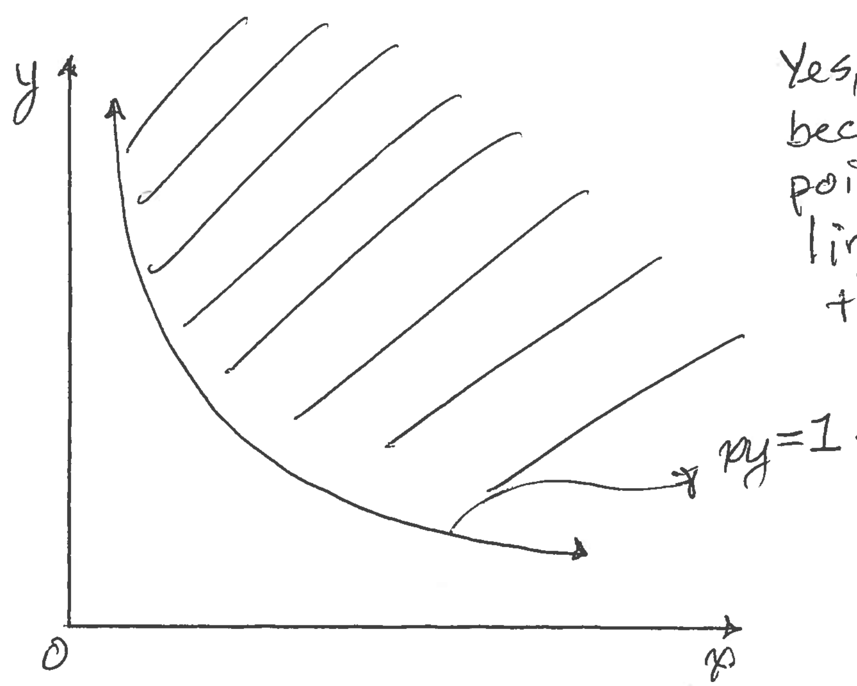
$LPM_2 = |H| = 80 - 64 = 16$

Since both LPM's are positive, the H matrix is positive definite.

Therefore the stationary point represents a local minimum.

(Actually, since the LPM's have the same signs regardless of the point at which they are evaluated, the function must be globally convex. Therefore the stationary point represents a global minimum.)

3. (X) Sketch the graph of the set $\{(x, y) \mid xy \geq 1, x > 0, y > 0\}$. Is this set convex? Why?



Yes, it is convex because, for any two points in the set, the line segment joining the two points is also in the set.

4. a.) Revising the notation: $f(x_1, x_2) = 2x_1^2 - x_1 x_2 + x_2^2$

Consider any two points: $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and $\hat{x} = (\hat{x}_1, \hat{x}_2)$.

The function is concave if $\lambda f(\bar{x}) + (1-\lambda) f(\hat{x}) \leq f(\lambda \bar{x} + (1-\lambda)\hat{x})$

$f(\bar{x}) = 2\bar{x}_1^2 - \bar{x}_1 \bar{x}_2 + \bar{x}_2^2$

$f(\hat{x}) = 2\hat{x}_1^2 - \hat{x}_1 \hat{x}_2 + \hat{x}_2^2$

$\lambda \bar{x} + (1-\lambda)\hat{x} = (\lambda \bar{x}_1 + (1-\lambda)\hat{x}_1 , \lambda \bar{x}_2 + (1-\lambda)\hat{x}_2)$

Then the right-hand side (RHS) of the formula becomes:

$$f(\lambda \bar{x} + (1-\lambda)\hat{x}) = 2[\lambda \bar{x}_1 + (1-\lambda)\hat{x}_1]^2 - (\lambda \bar{x}_1 + (1-\lambda)\hat{x}_1)(\lambda \bar{x}_2 + (1-\lambda)\hat{x}_2) + [\lambda \bar{x}_2 + (1-\lambda)\hat{x}_2]^2$$

And the left-hand side (LHS) of the formula becomes:

$\lambda(2\bar{x}_1^2 - \bar{x}_1 \bar{x}_2 + \bar{x}_2^2) + (1-\lambda)(2\hat{x}_1^2 - \hat{x}_1 \hat{x}_2 + \hat{x}_2^2)$.

So we have the indeterminate relation LHS ? RHS.

Now we must manipulate the LHS and RHS in order to determine the nature of the relation.

If LHS $\leq$ RHS, then the function is concave.

If LHS $<$ RHS, then the function is strictly concave.

If LHS $\geq$ RHS, then the function is convex.

If LHS $>$ RHS, then the function is strictly convex.

b.) If the function is concave, then the stationary point is a global maximum.

If the function is strictly concave, then the stationary point is a unique global maximum.

If the function is convex, then the stationary point is a global minimum.

If the function is strictly convex, then the stationary point is a unique global minimum.

5. Consider the equation $x^2 + 3xy + 2yz + y^2 + z^2 - 11 = 0$.

a.) First note that the equation has the form $F(x, y, z) = 0$, where $F(x, y, z) = x^2 + 3xy + 2yz + y^2 + z^2 - 11$.

Now check that the three conditions of the Implicit Function Theorem are satisfied.

First, does F have continuous partial derivatives?

$F_x = 2x + 3y$ $\qquad$ $F_y = 3x + 2z + 2y$ $\qquad$ $F_z = 2y + 2z$

All three derivatives are continuous.

Second, is the equation satisfied at the point? Yes.

Third, is $F_z \neq 0$ at the point? $F_z(1,2,0) = 2(2) + 2(0) = 4$

Since all three conditions are satisfied, it follows that $z = f(x, y)$ is implicitly defined at the given point.

b.) $$\frac{dz}{dx} = -\frac{F_x(1,2,0)}{F_z(1,2,0)} = -\frac{2(1) + 3(2)}{4} = -2$$