1. (3 pts.) (multiple-choice question) Find the limit

$$\lim_{x\to\infty}(xe^{1/x}-x) \qquad \infty - \infty$$

$$\lim_{x\to\infty} x(e^{1/x}-1) = \lim_{x\to\infty} \frac{e^{1/x}-1}{1/x}$$

$$\overset{0/0}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{-\frac{1}{x^2}e^{1/x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} e^{1/x} = 1$$

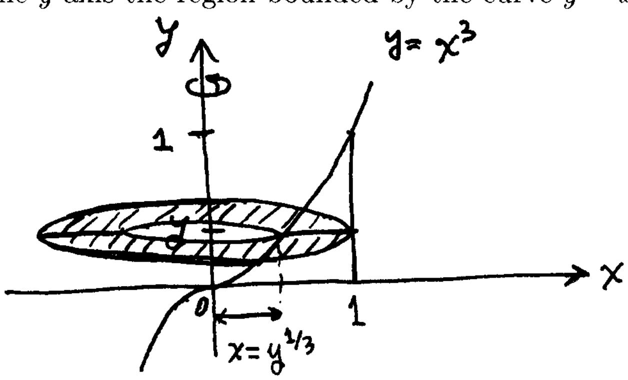
Select the correct answer.

A) 0 B) ∞

C) $-\infty$ D) 1 E) -1

Write the (capital) letter of the answer in this box ⟶ 1. $\boxed{\text{D)}}$

2. (10 pts.) Use the **disk/washer method** to find the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = x^3$, the line $x = 1$, and the x-axis.



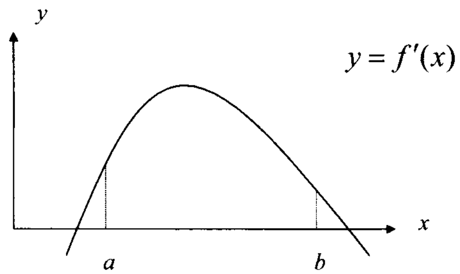
$$V = \int_0^1 A(y)\,dy$$

$$A(y) = \pi(1^2) - \pi(y^{1/3})^2$$
$$= \pi(1 - y^{2/3})$$

$$\Rightarrow \quad V = \pi\int_0^1 (1-y^{2/3})\,dy = \pi\left[y - y^{5/3}\tfrac{3}{5}\right]_0^1 = \pi\left(1-\tfrac{3}{5}\right)$$
$$= \tfrac{2}{5}\pi$$

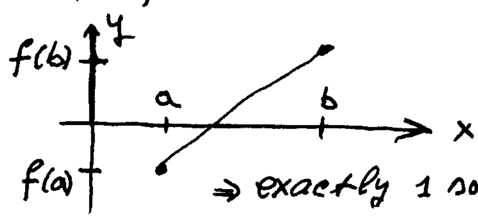
answer: $\boxed{\frac{2}{5}\pi}$

3. (3 pts.) (multiple-choice question) Suppose f is a differentiable function such that $f(a) < 0$ and $f(b) > 0$, and the graph of the derivative of f being given in the figure. What can be said about the number of solutions of the equation $f(x) = 0$ on the interval (a, b)?



$f'(x) > 0$ on (a,b)

$\Rightarrow$ f incr. on (a,b).



$\Rightarrow$ exactly 1 sol'n.

Select the correct answer.

A) $f(x) = 0$ does not have a solution on (a, b).
B) $f(x) = 0$ has exactly one solution on (a, b).
C) $f(x) = 0$ has exactly two solutions on (a, b).
D) $f(x) = 0$ has solutions on (a, b), but the exact number of solutions can not be determined from the given information.
E) The given information is insufficient for drawing any conclusion about solutions of $f(x) = 0$ on (a, b).

Write the (capital) letter of the answer in this box ⟶ 3. B)

4. (5 pts.) Find the critical numbers of the function $f(x) = x^{1/3} - x^{-2/3}$.

Domain of f: $x \neq 0$

$$f'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{1}{3}x^{-5/3}(x+2)$$

$f'(x) = 0 \Rightarrow x = -2$

f' DNE at $x = 0$, but $x = 0$ not in domain of f.

Method 2: $f(x) = x^{1/3} - \dfrac{1}{x^{2/3}} = \dfrac{x-1}{x^{2/3}}$.

$$f'(x) = \frac{x^{2/3} - \frac{2}{3}x^{-1/3}(x-1)}{x^{4/3}}$$

$$= \frac{x^{-1/3}\left(x - \frac{2}{3}x + \frac{2}{3}\right)}{x^{4/3}} = \frac{\frac{1}{3}x + \frac{2}{3}}{x^{5/3}}, \quad f'(x) = 0 \Rightarrow x = -2$$

answer: $x = -2$

5. (3 pts.) (multiple-choice question) Consider the function $f(x) = (x^2 - 3x - 24)\sqrt{x+1}$ with $f'(x) = \dfrac{5(x+2)(x-3)}{2\sqrt{x+1}}$. Find the absolute maximum and minimum values of f on the interval $[-1, 2]$.

$f'(x) = 0 \Rightarrow x = -2, 3$ not in $(-1, 2)$

$f'(x)$ DNE at $x = -1$ not in $(-1, 2)$

End pts. $f(-1) = 0$, $f(2) = (4-6-24)\sqrt{3} = -26\sqrt{3}$

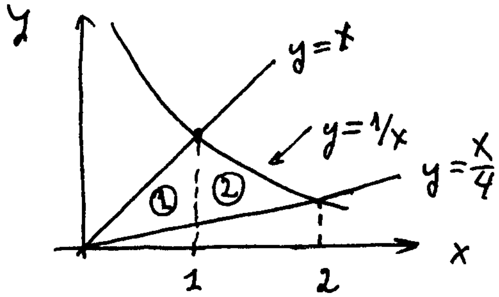
Select the correct answer.

A) 0 and -48 **B)** 0 and $-26\sqrt{3}$

C) 0 and $-26\sqrt{2}$ **D)** $-26\sqrt{2}$ and -48 **E)** $-26\sqrt{3}$ and -48

Write the (capital) letter of the answer in this box ⟶ 5. $\boxed{\text{B)}}$

6. (10 pts.) Find the area of the region bounded by the curve $y = 1/x$ and the lines $y = x$ and $y = \frac{1}{4}x$ with $x > 0$.



Intersections: $x = 1/x \Rightarrow x = 1, y = 1$

$\frac{1}{4}x = 1/x \Rightarrow x = 2, y = \frac{1}{2}$

$$A = \int_0^1 \left(x - \frac{x}{4}\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{x}{4}\right) dx$$

$$= \left[\frac{3}{8}x^2\right]_0^1 + \left[\ln x - \frac{x^2}{8}\right]_1^2 = \frac{3}{8} + \left(\ln 2 - \frac{1}{2} - 0 + \frac{1}{8}\right)$$

$$= \frac{4}{8} + \ln 2 - \frac{1}{2}$$

answer: $\boxed{\ln 2}$

$$= \ln 2$$

7. (3 pts.)(multiple-choice question) Find a function $f(x)$ such that

$$\int_1^x tf(t)\,dt - \frac{1}{4} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \qquad \text{for } x > 0.$$

FTC 1

$\frac{d}{dx}$: $x f(x) = x \ln x + \frac{1}{2} x^2/x - \frac{x}{2}$

$= x \ln x$

$\Rightarrow f(x) = \ln x$.

Select the correct answer.

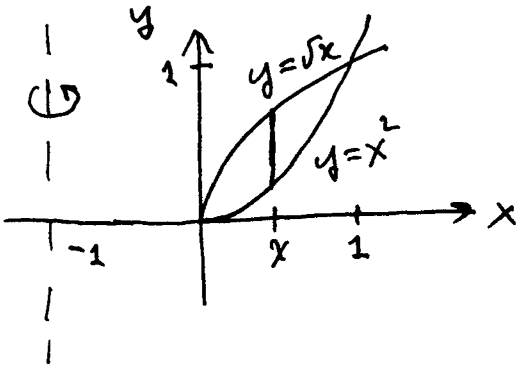
A) $f(x) = x \ln x + \frac{1}{4}$ **B)** $f(x) = \ln x$

C) $f(x) = x$ **D)** $f(x) = x + \frac{1}{4}$

E) $f(x) = \frac{1}{4}x$

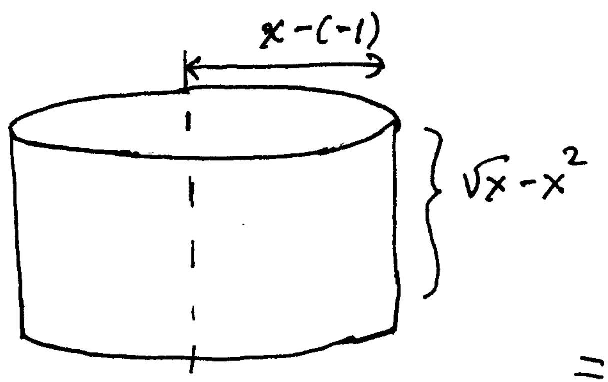
Write the (capital) letter of the answer in this box ⟶ 7. $\boxed{\text{B})}$

8. (10 pts.) Use the **method of cylindrical shells** to find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ about the line $x = -1$.



$$V = \int_0^1 S(x)\,dx$$

$$S(x) = 2\pi(x+1)(\sqrt{x} - x^2)$$
$$= 2\pi(x^{3/2} - x^3 + x^{1/2} - x^2)$$



$$V = 2\pi\left[x^{5/2}\frac{2}{5} - x^4\frac{1}{4} + x^{3/2}\frac{2}{3} - x^3/3\right]_0^1$$

$$= 2\pi\left(2/5 - 1/4 + 2/3 - 1/3\right)$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{4} + 1/3\right)$$

$$= \frac{2\pi}{60}(24 - 15 + 20)$$

answer: $\boxed{29\pi/30}$

9. (10 pts.) Evaluate the following integrals:

(a) $\int \frac{x^2}{\sqrt{1-x}}\,dx$

Let $u = 1-x \Rightarrow du = -dx$

$x = 1-u \Rightarrow x^2 = (1-u)^2$

$\Rightarrow \int \frac{1-2u+u^2}{u^{1/2}}(-1)du = -\int (u^{-1/2} - 2u^{1/2} + u^{3/2})\,du = -(2u^{1/2} - \frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2}) + c$

$= -2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + c$

$= -\frac{2}{15}(3x^2+4x+8)\sqrt{1-x} + c$ (optional)

answer: $\boxed{}$

(b) $\int_0^{\pi/3} \sec^3 x \, \tan x \, dx$

Let $u = \sec x \Rightarrow du = \sec x \tan x \, dx$.

$x=0 \Rightarrow u=1$, $x = \pi/3 \Rightarrow u = \frac{1}{1/2} = 2$

$\int_1^2 u^2 du = \frac{1}{3}u^3\Big]_1^2 = \frac{1}{3}(8-1) = \frac{7}{3}$

Method 2: $\int_0^{\pi/3} \frac{\sin x}{\cos^4 x} dx$ let $u = \cos x$, $du = -\sin x\, dx$

$-\int_1^{1/2} u^{-4} du = -\left[u^{-3}\left(-\frac{1}{3}\right)\right]_1^{1/2} = \frac{1}{3}(8-1)$

$= 7/3$

answer: $\boxed{7/3}$

10. (10 pts.) Evaluate the following integrals:

(a) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}(1+\sin^2\sqrt{x})}\,dx$

let $u = \sin\sqrt{x} \Rightarrow du = (\cos\sqrt{x})\frac{1}{2\sqrt{x}}dx$

$$\int \frac{2du}{1+u^2} = 2\tan^{-1}u + c$$

$$= 2\tan^{-1}(\sin\sqrt{x}) + c$$

answer: $\boxed{2\tan^{-1}(\sin\sqrt{x})+c}$

(b) $\displaystyle\int_{-2}^{1} (x - 3|x|)\,dx$

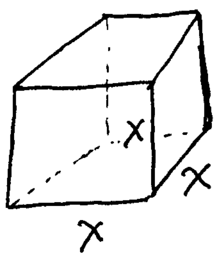
$$x-3|x| = \begin{cases} x-3x, & \text{if } x \geq 0 \\ x+3x, & \text{if } x<0 \end{cases} \Rightarrow x-3|x| = \begin{cases} -2x, & \text{if } x \geq 0 \\ 4x, & \text{if } x<0 \end{cases}$$

$$\int_{-2}^{0} 4x\,dx + \int_{0}^{1} -2x\,dx = \left[2x^2\right]_{-2}^{0} + \left[-x^2\right]_{0}^{1}$$

$$= -8 - 1 = -9$$

answer: $\boxed{-9}$

11. (8 pts.) The volume of a **cube** is increasing at a rate of 30 cm^3/min. How fast is the surface area increasing when the length of an edge is 6 cm.



$V = x^3$

$S = 6x^2$

Given	Req
$\frac{dV}{dt} = 30$	$\frac{dS}{dt} = ?$ when $x = 6$

Method 1

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow 30 = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{10}{x^2}$$

$$\frac{dS}{dt} = 12x \frac{dx}{dt} = 12x \left(\frac{10}{x^2}\right) = \frac{120}{x} \Rightarrow \left.\frac{dS}{dt}\right|_{x=6} = \frac{120}{6} = 20$$

Method 2

$$x = V^{1/3}$$

$$\Rightarrow S = 6V^{2/3} \Rightarrow \frac{dS}{dt} = 4V^{-1/3}\frac{dV}{dt} = 120V^{-1/3} \overset{V=x^3}{=} \frac{120}{x}$$

$$\Rightarrow \left.\frac{dS}{dt}\right|_{x=6} = \frac{120}{6} = 20$$

answer: $\boxed{20 \frac{cm^2}{min}}$

12. (10 pts.) Consider the function:

$$f(x) = \frac{x^3+x^2-1}{x^2-1} \quad \text{with} \quad f'(x) = \frac{x^2(x^2-3)}{(x^2-1)^2} \quad \text{and} \quad f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$$

(a) Find all the asymptotes of the graph of f.

$$\begin{array}{r|l} & x+1 \\ \hline x^2-1 & x^3+x^2-1 \\ & x^3-x \\ \hline & x^2+x-1 \\ & x^2-1 \\ \hline & x \end{array}$$

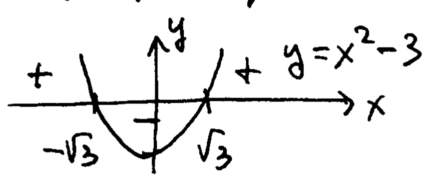
$\Rightarrow y = x+1$ S.A.

$\lim_{x\to 1} f(x)$ type $1/0 \Rightarrow x=1$ VA.

$\lim_{x\to -1} f(x)$ type $-1/0 \Rightarrow x=-1$ VA.

(b) Find the local (or relative) maximum and minimum values of f.

$\operatorname{sgn}(f') = \operatorname{sgn}(x^2-3)$



$f(\sqrt{3}) = \dfrac{3\sqrt{3}+3-1}{3-1} = \dfrac{3\sqrt{3}+2}{2}$ loc. min.

$f(-\sqrt{3}) = \dfrac{-3\sqrt{3}+3-1}{3-1} = \dfrac{2-3\sqrt{3}}{2}$ loc max.

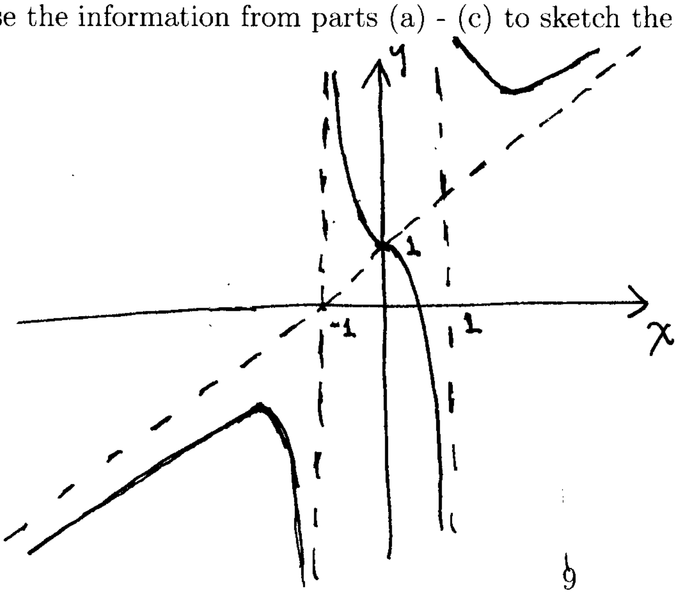
(c) Find the inflection points of the graph of f.

x	$-\infty$		-1		0		1		∞
x	— —		—	—	0	+ + +	+	+ +	
$(x^2-1)^3$	+ + +		0	— — —			0	+ + +	
f''	— — —		|	+ +	0	— —	|	+ + +	
f	$\frown$		|	$\smile$	IP	$\frown$	|	$\smile$	

$f(0) = 1$

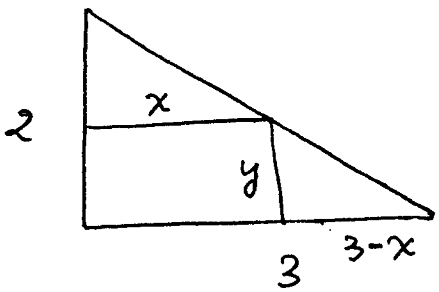
$\Rightarrow P(0,1)$ is I.P.

(e) Use the information from parts (a) - (c) to sketch the graph of f.



13. (10 pts.) Find the area of the largest rectangle that can be inscribed in a right triangle with legs (i.e. the sides that are perpendicular) of length 2 *cm* and 3 *cm* if the two sides of the rectangle lie along the legs.

Method 1:



$A = xy$

$\frac{y}{2} = \frac{3-x}{3} \Rightarrow y = \frac{2}{3}(3-x)$

$\Rightarrow A(x) = \frac{2}{3}(3x - x^2), \quad 0 \le x \le 3$

$A'(x) = \frac{2}{3}(3-2x) \qquad A'(x) = 0 \Rightarrow x = 3/2$

Test: closed interval method: $A(0) = A(3) = 0$, $A(3/2) = \frac{2}{3}\left(\frac{9}{2} - \frac{9}{4}\right) = 3/2$ (max)

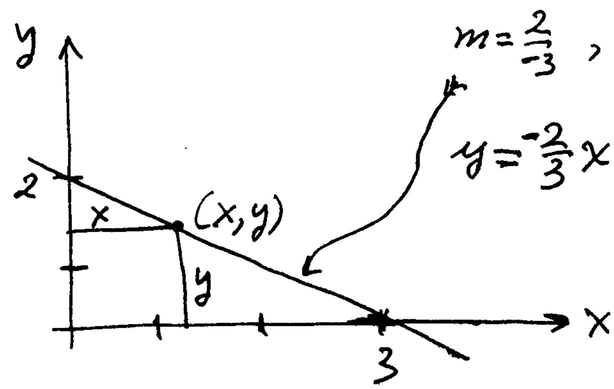
or 1st deriv. test:

x	0	$3/2$	3
$3-2x$	$+++$	0	$--$
A	$\nearrow\nearrow$	max	$\searrow\searrow$

$A(3/2)$ max.

or 2nd deriv test: $A''(x) = -4/3 < 0 \Rightarrow A(3/2)$ max.

Method 2:



$m = \frac{2}{-3}$, y-int. $= 2$

$y = \frac{-2}{3}x + 2 = \frac{2}{3}(3-x)$ as in Method 1.

answer: $\boxed{\text{Area} = 3/2}$