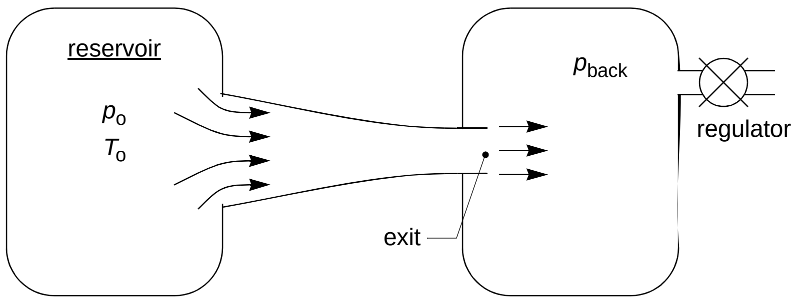
# 6. Applications of Isentropic Flow

Until now, we have used one-dimensional flow relations to relate one point in an isentropic flow to another point in the same flow. Now we will apply these relations to solve an entire flowfield though a nozzle and then examine the application of nozzles to rocket propulsion.

## 6.1 Converging Nozzle

Consider a converging duct or nozzle that is fed a flow of gas from a reservoir with stagnation conditions $p_o$, $T_o$. We will assume these stagnation conditions are *fixed* for the following discussion. The flow from the nozzle exhausts in a chamber with a controlled back pressure, $p_{back}$. The back pressure is controlled by means of a regulator.

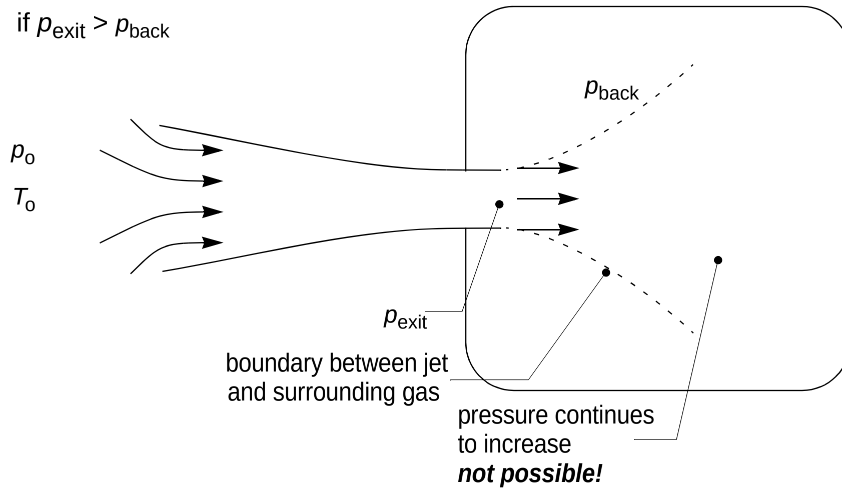



We can now consider the possible flows:

If $p_o = p_{back}$, then obviously there is no flow (no pressure difference to drive flow).

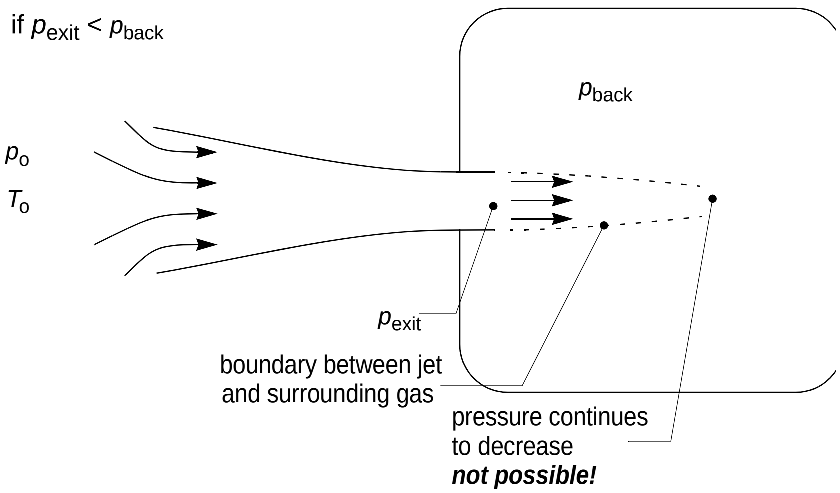
If $p_o > p_{back}$, then there will be a flow. The exit flow could be subsonic or sonic, but not supersonic since there is no converging-diverging section to bring the flow smoothly from subsonic to supersonic flow.

If the exit flow is subsonic, then the *exit pressure must match the back pressure*: $p_{exit} = p_{back}$. This is because, in subsonic flow, information can be communicated upstream and the flow will adjust itself to match the back pressure. Consider what would happen if the exit pressure did *not* match the back pressure: if the exit pressure was greater, then the higher pressure flow would push the streamtube outward, resulting in a diverging area of the jet as it expands into the chamber:

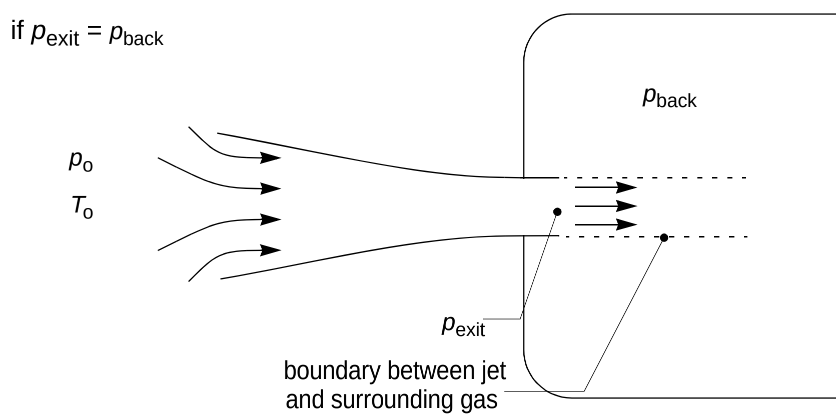



In this case, the flow would diverge. A diverging subsonic flow tends to decelerate, increasing in pressure. Thus, the pressure in the jet would increase even more than the ambient pressure, pushing the streamtube further out, increasing the pressure still further. This is clearly non-physical.

Likewise, if the exit pressure was less than ambient, the ambient pressure would squeeze the jet inward, resulting in the jet lowering pressure further as it flows into a converging area. This is also non-physical.




Thus, the only subsonic exit flow that is consistent is if *the exit flow matches the back pressure exactly.* Again, this makes sense because, for a subsonic flow, the flow in the nozzle can accommodate itself to match the conditions in the receiving chamber.*




*Of course, the jet will initially decelerate due to viscous drag and entrainment of the surrounding gas into the jet. We cannot treat this flow with steady, one-dimensional isentropic flow.

For subsonic exit flow, since we have now established that the exit flow is at the same pressure as the back pressure, we can solve for the exit Mach number using:

$$\frac{p_o}{p_{exit}} = \left(1 + \frac{\gamma - 1}{2} M_{exit}^2\right)^{\frac{\gamma}{\gamma - 1}}$$

Knowing the exit Mach number and exit area, we can solve for any other point in the nozzle. Thus, the back pressure becomes the parameter that controls the entire nozzle flow.

What if we decrease the back pressure to the point where the exit flow is sonic. This will occur when $p_{back} = p_{exit} = p^*$. Recall from Section 5.3:

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}} = 0.5283 \text{ for } \gamma = 1.4$$

Thus for a back pressure of $p_{back} = 0.5283\, p_o$, the flow at the exit plane will be at Mach 1.

What if we decrease the back pressure below $p^*$? Since the flow is sonic at the exit of the nozzle, the fact that the back pressure has been lowered cannot be communicated into the nozzle. In fact, the mass flow through the nozzle is choked, so further decreasing the back pressure does not increase mass flow or alter the flow in the nozzle in any way.

We can summarize these cases as follow:

- $p_{back} > p^*$ $\Rightarrow$ $M_{exit} < 1$

  $\Rightarrow$ Pressure profile in nozzle depends on $p_{back}$

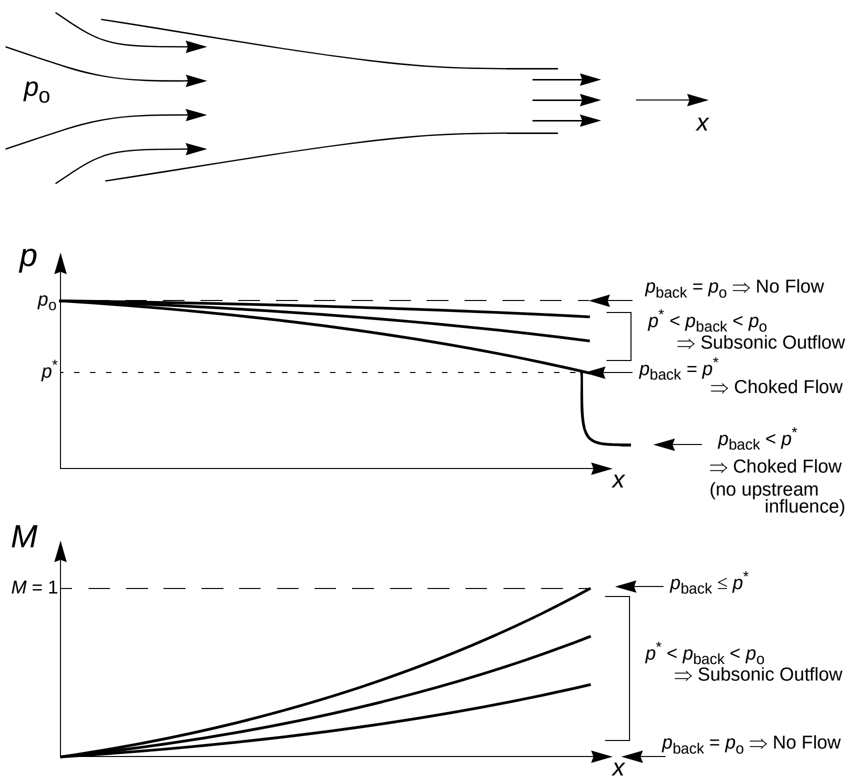
  $\Rightarrow$ $\dot{m}$ depends on $p_{back}$

- $p_{\text{back}} \le p^*$ $\Rightarrow$ $M_{\text{exit}} = 1$

  $\Rightarrow$ Pressure profile in nozzle independent of $p_{\text{back}}$

  $\Rightarrow$ $\dot{m}$ independent of $p_{\text{back}}$

We can represent these possibilities graphically:

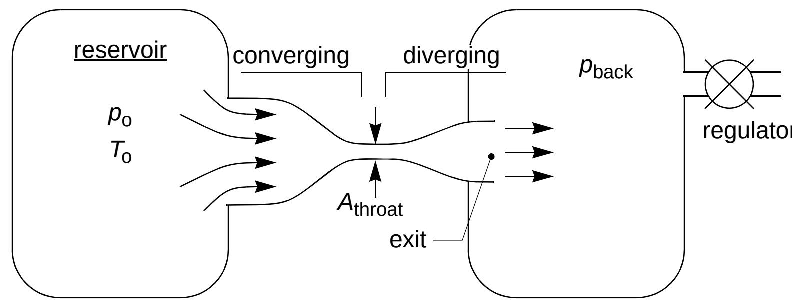



Another way we can express the condition for sonic outflow is if $p_o > 1.89 p_{back}$ for $\gamma = 1.4$. Thus, anytime a reservoir is approximately twice the ambient pressure or greater, the discharge of gas from the reservoir will choke, with sonic conditions at the exit of the nozzle.

Consider a bicycle tire, which is typically inflated to 3-4 atm. When we depress the pin to allow the air to escape to ambient pressure (1 atm), does the flow actually choke and reach sonic flow at the exit? ***Yes***, it does!

## 6.2 Converging-Diverging Nozzle

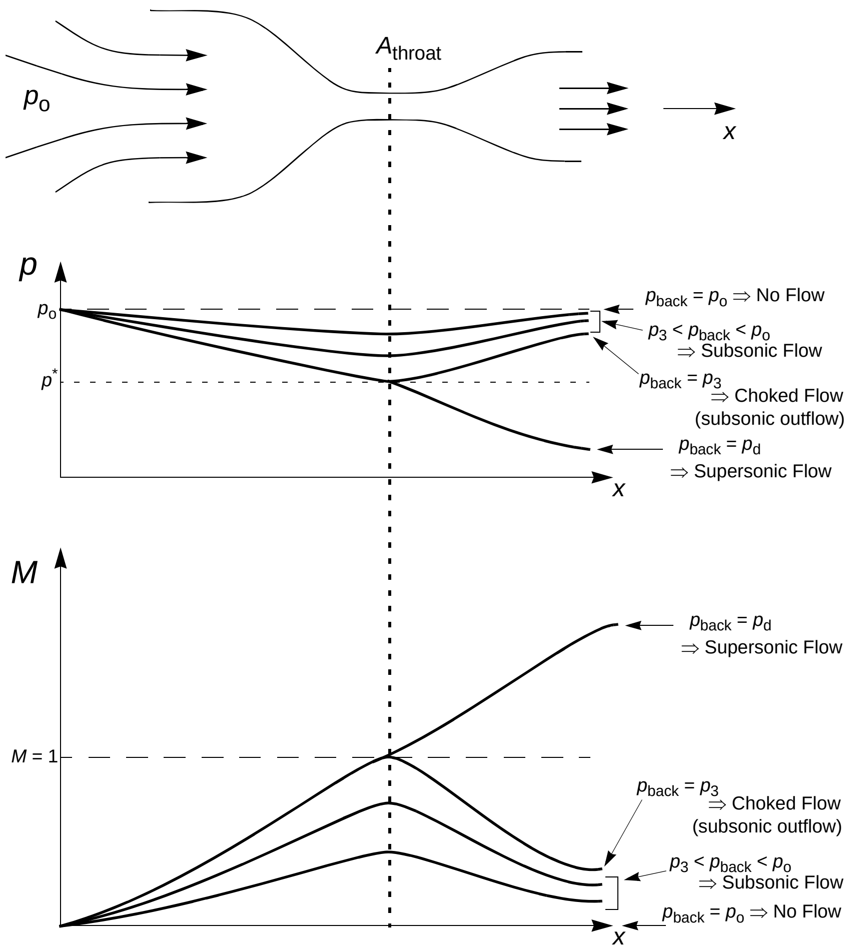
We will now consider the case of a converging-diverging nozzle with a throat of minimum area $A_{\text{throat}}$:




The analysis is similar to that for a converging nozzle: as $p_{\text{back}}$ is made less than $p_o$, a flow will be induced through the nozzle. The flow will accelerate in the converging section and decelerate in the diverging section (as long as it does not reach sonic at the throat). As $p_{\text{back}}$ decreases, the velocity and mass flow through the nozzle increases, until sonic conditions are reached at the throat ($M_{\text{throat}} = 1$). At this point, the flow is choked and lowering the back pressure does not increase the mass flow further. As the flow is sonic at the throat, there are two possible solutions in the diverging section: subsonic or supersonic. The subsonic solution brings the flow back down the same thermodynamic path as in the converging solution. For the supersonic solution, the flow continues to accelerate and the pressure decreases, until the flow exits at $p_d$.

Which solution actually occurs depends on the back pressure: $p_d$ refers to the *design* case of the nozzle, where the nozzle was designed to produce supersonic flow at this condition (if subsonic flow was desired, a simple converging nozzle would have been used instead). The supersonic solution will only be realized if the back pressure has the correct value to force the supersonic solution.

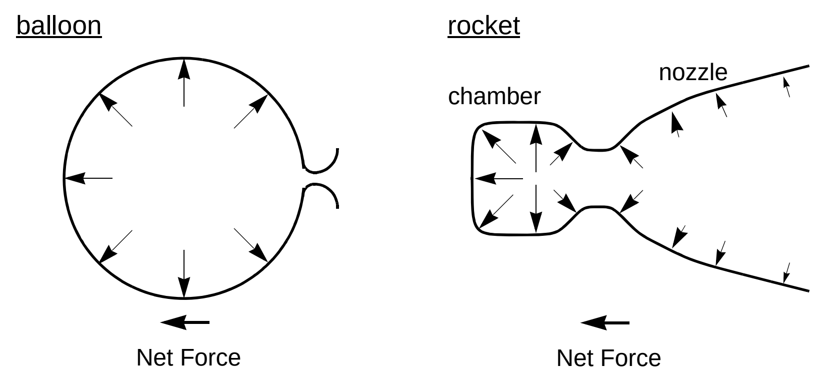
These cases are illustrated here:




An obvious question is: what happens for back pressures between $p_3$ and $p_d$? For this range of back pressure, no isentropic solution to the converging-diverging nozzle is possible. We will have to wait until Chapter 8, where we will re-examine the converging-diverging nozzle with the possibility of shock waves in the diverging section, in order to treat this range of back pressures. Another question is: what happens at back pressures less than $p_d$? In this case, the exit flow is isentropic, but can no longer be treated as one-dimensional. Analysis of this case will have to await until Chapter 14.

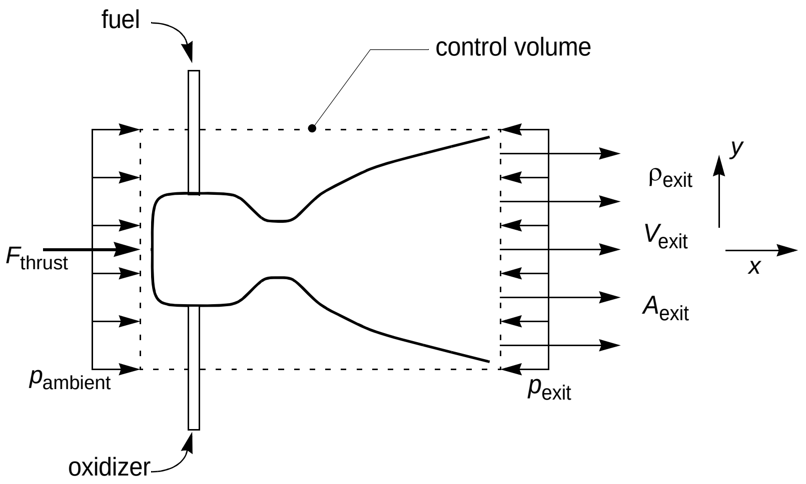
## 6.3 Rocket Nozzles

An important application of one-dimensional isentropic flow is the nozzle as found on almost all aerospace propulsion devices. There are different ways we can view the thrust-generating mechanism of a rocket engine. One way is as a pressure imbalance generated inside the chamber, in analogy to a balloon:




In the rocket, combustion of fuel and oxidizer in the combustion chamber generates high pressure gas. Similar to the balloon, the fact that there is an opening in the chamber generates a force imbalance that results in a net force on the rocket, i.e., thrust. By using a nozzle, additional thrust can be generated as the high pressure products expand out to ambient conditions.

To actually compute the rocket thrust by integrating the pressure acting on the rocket surfaces is very difficult.[*] We would first need to solve the flow though the entire engine, and then integrate the pressure distribution over the internal geometry of the engine, which can be quite complex. Aerospace engineers instead tend to put the rocket inside a conceptual "black box" or control volume:




Note that the thrust is drawn as the force acting *on* the control volume (a control volume is similar to a free body diagram from mechanics). The force of the rocket on the rest of the vehicle (or the thrust stand) will be equal and opposite to the force on the rocket engine shown here. Also note that the ambient pressure $p_{amb}$ acts on the entire control volume, except at the exit plane, where $p_{exit}$ acts. The fuel and oxidizer inlets are shown here acting in the *y*-direction, so they do not contribute to the *x*-momentum equation, but in general the fuel and oxidizer are injected in liquid form at relatively low velocity, so they do not have a significant effect on the momentum equation even if they are injected in the *x*-direction. For a solid-fueled rocket, the propellant gases evolve directly from the walls of the combustion chamber, so again, the propellant feed does not enter the momentum equation.

Applying the momentum equation:

$$\Sigma\vec{\boldsymbol{F}} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\rho\vec{\boldsymbol{V}}dV - \int_{CS}\vec{\boldsymbol{V}}(\rho\vec{\boldsymbol{V}}\cdot d\vec{A})$$

to the *x*-direction for steady operation of a rocket with one-dimensional exit flow yields:

$$F_{\mathrm{thrust}} = \dot{m}V_{exit} + A_{exit}(p_{exit} - p_{amb})$$

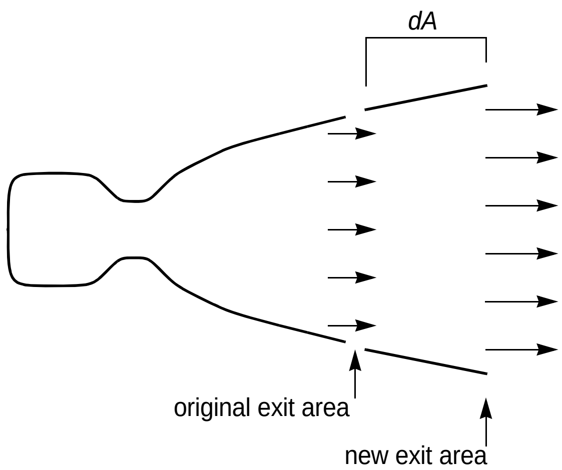
Note that there are *two* contributions to the thrust produced by the rocket: a momentum flux term $\dot{m}V_{exit}$ and a pressure difference term $A_{exit}(p_{exit} - p_{amb})$. Further, since in general the exit flow from a rocket nozzle will be supersonic, the exit flow pressure does not need to necessarily match the ambient pressure. Thus, when viewed from the control volume point of view, a rocket generates thrust from both the momentum flux through the control volume and the net pressure pushing on the control volume.

### 6.3.1 Conditions for Optimal Thrust for Given Ambient Pressure

Which one of these terms will be dominant? Or, expressed another way, which is the optimal way to operate a rocket nozzle: with a high exit pressure or a large momentum flux? Consider a rocket with fixed chamber conditions and fixed $\dot{m}$.

Does adding an additional length of nozzle which increases the area further by an amount *dA* increase or decrease the net thrust? Of course, increasing the exit area will, for a supersonic flow, increase the exit velocity, but it will also decrease the exit pressure. Thus, the $\dot{m}V_{exit}$ will increase, but the $A_{exit}(p_{exit} - p_{amb})$ term may decrease (even though $A_{exit}$ is increasing) or even become negative if $p_{exit}$ drops below $p_{amb}$, creating a negative thrust or "suction" force. Thus, these two terms compete with each other, and it is not obvious which term will win out.

*Note that there will be instances where calculations requiring the complete pressure distribution inside the rocket will be required; performing a stress analysis on the rocket structure, for example.




In order to see how $F_{\text{thrust}}$ varies with the exit area, we can take a derivative:

$$\frac{\mathrm{d}F_{\text{thrust}}}{\mathrm{d}A_{\text{exit}}} = \dot{m}\frac{\mathrm{d}V_{exit}}{\mathrm{d}A_{\text{exit}}} + (p_{exit} - p_{amb}) + A_{\text{exit}}\frac{\mathrm{d}p_{exit}}{\mathrm{d}A_{\text{exit}}}$$

Note that $\dot{m}$ and $p_{\text{amb}}$ are assumed constant. Expanding $\dot{m} = \rho VA$, we can write this as:

$$\frac{\mathrm{d}F_{\text{thrust}}}{\mathrm{d}A_{\text{exit}}} = A_{\text{exit}}\left(\frac{\mathrm{d}p_{exit}}{\mathrm{d}A_{\text{exit}}} + \rho_{\text{exit}}V_{\text{exit}}\frac{\mathrm{d}V_{exit}}{\mathrm{d}A_{\text{exit}}}\right) + (p_{exit} - p_{amb})$$

Recall the differential momentum equation for inviscid flow:

$$\frac{dp}{\rho} + VdV = 0$$

we can show that the first term in parentheses disappears to leave:

$$\frac{\mathrm{d}F_{\text{thrust}}}{\mathrm{d}A_{\text{exit}}} = (p_{exit} - p_{amb})$$

Thus, as long as the exit pressure is greater than the ambient pressure, we can increase thrust by increasing the exit area. If the exit pressure is less than the ambient pressure, then increasing the exit area results in lower thrust. The maximum thrust condition, for fixed chamber conditions and a fixed ambient pressure, (called "optimal conditions") will be when

$$\frac{\mathrm{d}F_{\text{thrust}}}{\mathrm{d}A_{\text{exit}}} = 0 \Rightarrow p_{exit} = p_{amb}$$

Thus, the *optimal operating conditions for a rocket nozzle occur when the exit pressure matches the ambient pressure*, and all of the thrust results from momentum flux and no thrust results from the pressure term.* Anytime the exit flow is at a higher pressure, we can extract more thrust by accelerating the flow to a higher velocity and lowering the exit pressure. If the exit pressure is less than the ambient pressure, however, the negative thrust or "suction" drag will more than offset the larger exit velocity, and greater thrust can be achieve by reducing the exit velocity.

### 6.3.2 Maximum Thrust for Given Reservoir Conditions

We have seen that the optimal way to operate a rocket nozzle at a given altitude (i.e., a fixed back pressure) is by having the exit pressure match the back pressure. The *maximum thrust* we can obtain from a given combustion chamber (i.e., reservoir conditions) will be when the rocket operates in vacuum and expands the flow all the way to zero pressure. A rocket operating in outer space with a very large nozzle area ratio (effectively infinite area ratio) can realize this maximum amount of thrust; there is no way to obtain more thrust from the given combustion chamber conditions. To determine what the exit velocity will be in this case, we can examine the energy equation

$$\left(h + \frac{V^2}{2}\right) = h_o = \text{constant}$$

*Note that while our discussion assumed supersonic exit flow, this analysis did not make any assumptions regarding the flow exit conditions. Thus, the optimal case is when the exit flow matches the ambient pressure, regardless of whether it is subsonic or supersonic.

For a calorically perfect gas:

$$\left(c_p T + \frac{V^2}{2}\right) = c_p T_o$$

This relation applies everywhere in the nozzle. If we expand the flow all the way to vacuum, then there will be a complete conversion of thermal energy to kinetic energy, and static temperature will go to zero. Thus, the exit velocity in this case will be:

$$V_{max} = \sqrt{2c_p T_o}$$

Recalling that $c_p = R\frac{\gamma}{\gamma - 1}$, we can express $V_{max}$ as

$$V_{max} = \sqrt{\frac{2}{\gamma - 1}\gamma R T_o} = \sqrt{\frac{2}{\gamma - 1}}c_o = 2.236c_o \text{ (for } \gamma = 1.4\text{)}$$

This is the maximum velocity that can be created via a steady expansion with stagnation temperature $T_o$.* We see that it is simply a constant multiple of the initial speed of sound in the reservoir $c_o$. Since there is no pressure contribution to thrust when the exit flow is fully expanded to ambient vacuum, the thrust is simply

$$F_{\text{thrust}} = \dot{m}V_{max}$$

Aerospace engineers usually speak in terms of a measure of rocket engine performance called *Specific Impulse*, which is defined as thrust per kilogram/sec of propellant consumed:

$$I_{sp} = \frac{F_{\text{thrust}}}{\dot{m}}$$

*Note that it is possible to create greater velocity flow starting with the same stagnation conditions if an *unsteady* expansion is used. See Chapter 16 for details.

We can think of $I_{sp}$ as a measure of the "fuel efficiency" of a rocket: it is a measure of many Newtons of thrust are produced per kilogram of propellant consumed.* We can see that the maximum $I_{sp}$ is simply the maximum exit velocity:

$$(I_{sp})_{max} = V_{max} = \sqrt{\frac{2}{\gamma - 1}}c_o$$

Thus, the best rocket propellants are those with a large initial speed of sound $c_o$. From the expression for the speed of sound $c = \sqrt{\gamma\frac{\tilde{R}}{MW}T}$, we can see that the best propellant has a *large initial temperature* and a *low molecular weight.*

## 6.3.3 General Rocket Nozzle Equation

So far, we have examined the optimal cases where the nozzle exhaust matches the ambient pressure (or, in the case of maximum thrust, exhaust matches the ambient vacuum). In general, however, rockets will operate at less than optimal conditions. In this case, there will be a contribution due to the pressure at the nozzle exit plane, and we will have to consider the second term on the right hand side:

$$F_{\text{thrust}} = \dot{m}V_{exit} + A_{exit}(p_{exit} - p_{amb})$$

While we can solve for $V_{exit}$ using our relations for isentropic flow, it is convenient to express the exit velocity in terms of the stagnation conditions in the chamber. Starting with the energy equation for a perfect gas:

*Note that $I_{sp}$ has units [m/s]. Unfortunately, most rocket technology was developed with the Imperial systems of units, where $I_{sp}$ is defined as "pound force of thrust per pound mass of propellant consumed." This results in $I_{sp}$ having units of seconds in the Imperial units system. The two are related by: $(I_{sp})_{metric} = g_o(I_{sp})_{Imperial}$, where $g_o$ is the acceleration due to gravity ($g_o = 9.81$ m/s$^2$). Note that we use the value $g_o = 9.81$ m/s$^2$ for the Earth's surface, even when a rocket is operating in space or on a different planet.

$$V = \sqrt{2c_p(T_o - T)}$$

Or

$$V = \sqrt{\frac{2\gamma}{\gamma-1}RT_o\left(1 - \frac{T}{T_o}\right)}$$

Since the flow is isentropic, $\frac{T}{T_o} = \left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}}$. Using this:

$$V = \sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1 - \left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

Substituting this into the expression for thrust:

$$\boxed{F_{\text{thrust}} = \dot{m}\sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1 - \left(\frac{p_{exit}}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} + A_{exit}(p_{exit} - p_{amb})}$$

This equation applies to any rocket, even if not operating at optimal conditions. If it is operating at optimal conditions, $p_{exit} = p_{amb}$ and the relation reduces to:

$$(F_{\text{thrust}})_{optimal} = \dot{m}\sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1 - \left(\frac{p_{amb}}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

In this case, the $I_{sp} = \frac{F_{\text{thrust}}}{\dot{m}}$ is simply the exit velocity. If the rocket is operating in vacuum and expands full to zero pressure ($p_{exit} = 0$), then the maximum thrust can be achieved:

$$(F_{\text{thrust}})_{maximum} = \dot{m}\sqrt{\frac{2\gamma}{\gamma-1}RT_o}$$

which is the same result we obtained in the last section. Thus, the general rocket expression is valid for any rocket, and the relations for optimal performance and maximum thrust are "built in."

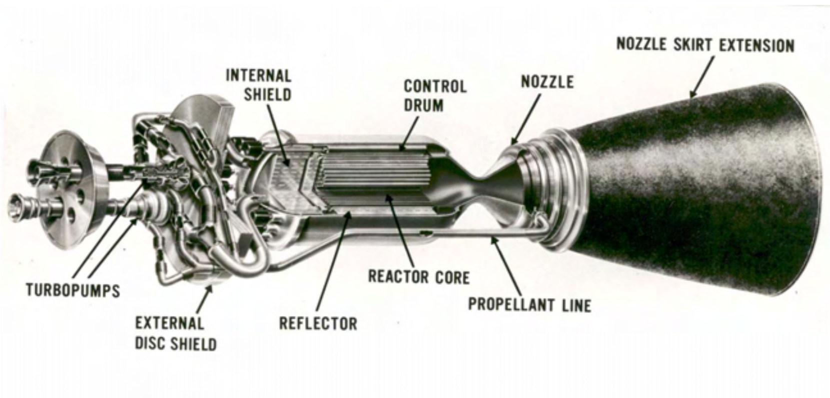
### 6.3.4 Numerical Example: Nuclear Thermal Rocket

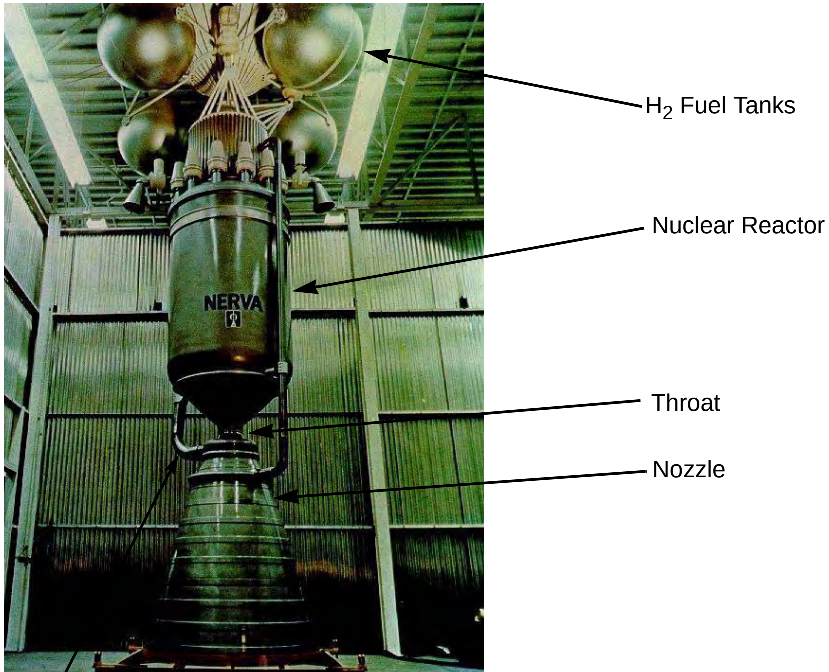
As we have seen in the prior sections, the best propellant for a rocket would have a low molecular weight and be heated to as high a temperature as possible. The lowest molecular weight substance is molecular hydrogen ($H_2$, $MW = 2$). To generate high temperatures, we could burn hydrogen with oxygen. This is a very energetic reaction: hydrogen releases more energy per mass when burned with oxygen than almost any other fuel, and the products can reach temperatures in excess of 3000 K. Unfortunately, oxygen is a heavy atom ($MW = 16$), so this will increase the average weight of the products of combustion (i.e., steam: gaseous $H_2O$, $MW = 18$) and lower the performance of a pure $H_2$ rocket.

In the 1960's, engineers began work on an engine that would use pure $H_2$ as the propellant. To heat the propellant to high temperature, a light weight nuclear reactor was designed and built. The temperature of the chamber was now only limited by the material constraints of the reactor and chamber walls. The NERVA (nuclear engine for rocket vehicle application) as shown below was actually test fired in Nevada.* This is a convenient rocket for us to study because the propellant used (pure hydrogen) can be well approximated as a calorically perfect gas with $\gamma = 1.4$. More conventional rockets using combustion to produce the propellant gas result in a chemically reacting flow that is not calorically perfect and $\gamma$ and $MW$ vary through the expansion process. Thus, they are not as amenable to the simple analysis we developed here.

*Note this was before the era of "Environmental Impact Statements."

NERVA Nuclear Rocket (1960's)




**Note:** propellant circulated around nozzle before injection into reactor in order to cool nozzle (regenerative cooling).

Assume that the chamber can tolerate temperatures of 3000 K and operates at a pressure of 30 atm. These conditions then define the stagnation conditions for the nozzle. We are now asked to...

(a) Find optimal $I_{sp}$ at sea level.

(b) If the same nozzle from (a) is operated in space, find the $I_{sp}$.

(c) Find the maximum $I_{sp}$ possible by operating in space.

(d) Find the $I_{sp}$ for expansion ratio of 500 in space.

We can find all of these via straightforward application of the rocket nozzle equation developed above.

<u>(a) Optimal $I_{sp}$ at Sea Level</u>

For optimal conditions at sea level, $p_{exit} = p_{amb}$ =1 atm. Using $T_o$ = 3000 K and $p_o$ = 30 atm, the rocket nozzle equation gives the $I_{sp}$ as:

$$(F_{\text{thrust}})_{optimal} = \dot{m}\sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1-\left(\frac{p_{amb}}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$


NERVA on its way to test range in Nevada. Note nozzle is pointed upward. Also note people for scale.

$$(I_{sp})_{optimal} = \sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1-\left(\frac{p_{amb}}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} = 7366 \text{ m/s}$$

This is a truly phenomenal value of $I_{sp}$. The Space Shuttle Main Engine (SSME) is the highest performance chemical rocket engine built, and it has an $I_{sp}$ of about 4100 m/s at sea level.

Although we did not need to determine it in order to compute $I_{sp}$, it is of interest to determine the exit Mach number and area ratio. Knowing $\frac{p_{exit}}{p_o}$, we can find the exit Mach number: $M_{exit} = 2.87$ and the area ratio $\frac{A}{A^*} = 3.74$.

(b) Optimized for Sea Level, Operated in Space

If we fix the area ratio from part (a) at $\frac{A}{A^*} = 3.74$ and bring the rocket to space, where the ambient pressure is $p_{amb} = 0$ atm*, what will the $I_{sp}$ be? Since we are assuming isentropic flow in the nozzle, the conditions at the nozzle exit are unchanged from (a) and the exit pressure will be $p_{exit} = 1$ atm. Thus, we now need to consider the second term on the right hand side of the rocket nozzle equation:

$$F_{\text{thrust}} = \dot{m}\sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1-\left(\frac{p_{exit}}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} + A_{exit}(p_{exit}-p_{amb})$$

Or:

*While space is not a perfect vacuum, it is very close. In fact, the "gas" in Low Earth Orbit is at such a low density, it can no longer be considered a continuum. Thus, from the perspective of compressible fluid dynamics, it is vacuum.

$$I_{sp} = \sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1-\left(\frac{p_{exit}}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \frac{A_{exit}}{\dot{m}}(p_{exit}-p_{amb})$$

Recall from Section 5.4 that the term $\frac{A}{\dot{m}}$ can be written as

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}}\frac{p_o}{\sqrt{T_o}}\frac{M}{\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

The $I_{sp}$ becomes:

$$I_{sp} = \sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1-\left(\frac{p_{exit}}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \sqrt{\frac{RT_o}{\gamma}}\frac{\left[1+\frac{\gamma-1}{2}M_{exit}^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}{M_{exit}}\frac{(p_{exit}-p_{amb})}{p_o}$$

$$I_{sp} = 7366 + 643 = 8009 \text{ m/s}$$

Thus, specific impulse increases to a value of $I_{sp} = 8006$ m/s when we operate the same rocket in space.

(c) Maximum $I_{sp}$ Operating in Space

Although the $I_{sp}$ we found in (b) is greater than in (a), we can obtain an even higher $I_{sp}$ if we expand the flow further to zero pressure:

$$(I_{sp})_{maximum} = \sqrt{\frac{2\gamma}{\gamma-1}RT_o} = 9343 \text{ m/s}$$

Thus, we can greatly increase the specific impulse by completely expanding the flow. This will necessitate using a nozzle of infinite area ratio. In practice, nozzles used in space typically have exit-to-throat area ratios in the range of 100 – 500, which results in the flow being very nearly fully expanded. Using larger area ratio nozzles can lower the actual $I_{sp}$, because a large nozzle must also be longer, and viscous drag of flow on the nozzle walls can no longer be neglected. The actual NERVA rocket was designed for use in space and would have used a nozzle area ratio of 500.

(d) Area Ratio of 500, Operated in Space

For an area ratio of $\frac{A}{A^*} = 500$, the exit Mach number is $M_{exit} = 9.83$. Thus, the exit pressure will be $p_{exit} = 7.9 \times 10^{-4}$ atm. Thus, the flow is nearly expanded to vacuum. The specific impulse is:

$$I_{sp} = \sqrt{\frac{2\gamma}{\gamma-1}RT_o\left[1-\left(\frac{p_{exit}}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \sqrt{\frac{RT_o}{\gamma}}\frac{\left[1+\frac{\gamma-1}{2}M_{exit}^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}{M_{exit}}\frac{(p_{exit}-p_{amb})}{p_o}$$

$I_{sp} = 9111 + 67 = 9178$ m/s. Thus, the $I_{sp}$ with this nozzle is about 2.5% below the maximum possible $I_{sp}$.

Summary

| | $A/A^*$ | $I_{sp}$ at Sea Level | $I_{sp}$ in Space |
|---|---|---|---|
| Optimized for Sea Level | 3.74 | 7366 m/s | 8009 m/s |
| Optimized for Space | ∞ | N/A* | 9343 m/s |
| Actual Nozzle | 500 | N/A* | 9178 m/s |

*If we operated a nozzle optimized for vacuum at sea level, we will see in Chapter 8 that the flow is so severely overexpanded a normal shock will appear in the nozzle, which means that the flow is no longer isentropic and our analysis is invalid. The $I_{sp}$ in these cases will be much less than the $I_{sp}$ of the nozzle optimized for operation at sea level.

We can see from this problem that, if we want to design a rocket that operates at both sea level and in space, we need to make a compromise on the performance of a nozzle: a nozzle optimized for sea level will perform poorly when operated in space, as compared to a nozzle optimized for operation in vacuum. This creates a unique challenge for single-stage launch vehicles, which must operate efficiently from sea level to space. This issue is discussed further in Section 6.3.5 below.

The outstanding performance of the Nuclear Thermal Rocket (NTR) has motivated engineers to reexamine the concept several times since the 1960's, but so far no NTR has actually flown. Recently, there has been considerable interest in a related concept: the Solar Thermal Rocket. This rocket would use large parabolic mirrors to focus sunlight into a combustion chamber and heat the propellant. This concept is currently more politically acceptable than NTR's, and a Solar Thermal Rocket will very likely be test flown in space in the next few years.

Artist's conception of a Solar Thermal Rocket operating in Low Earth Orbit. Inflated films of thin reflective material act as parabolic mirrors to focus sunlight into the "combustion" chamber, heating hydrogen propellant that is then fed into the nozzle.

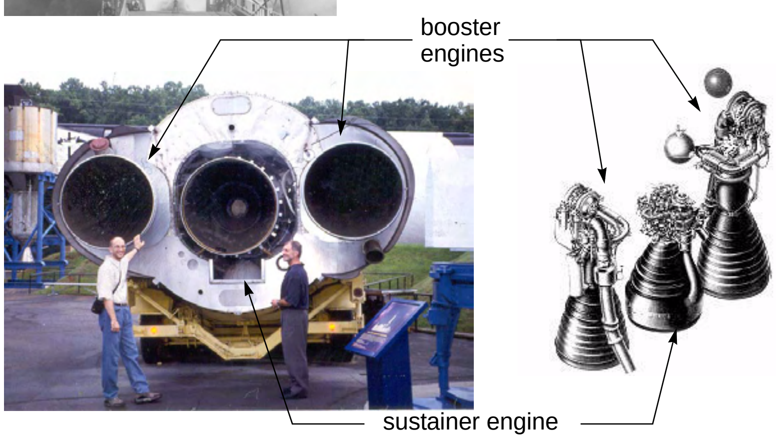
## 6.3.5 Achieving Optimal Nozzle Performance

The fact that exit pressure should match the ambient pressure for optimal operating conditions of a rocket nozzle creates an interesting challenge for launch vehicles that operate over a range of ambient pressures. For a missile or rocket stage operating over a narrow range in altitude, it is relatively straight forward to optimize the nozzle for fixed operating conditions. But consider the Space Shuttle Main Engine (SSME), which must operate from sea level

($p_{amb}$ = 1 atm) to Low Earth Orbit ($p_{amb}$ = essentially vacuum). How can a nozzle give optimal conditions over this range of ambient pressures? The convention answer is: “It can’t!” and instead aerospace engineers have to rely upon a compromise design that is neither optimal at sea level or in vacuum.

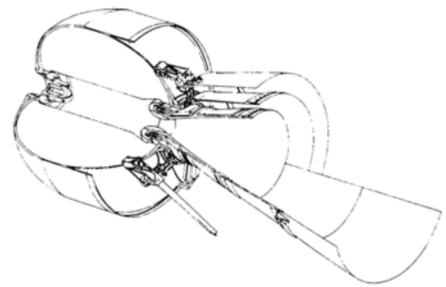
Reviewing the results of Section 6.3.4, the effect of matching or not matching ambient pressure may seem small, but aerospace engineers, “Live and die by the third significant digit of $I_{sp}$.”* Thus, there is considerable interest in designing a rocket nozzle that can provide near-optimal operating conditions for a varying back pressure.

The Atlas launch vehicle launched the first American into orbit in 1962.




One solution adopted by the Atlas launch vehicle, which launched the first American astronauts into orbit in the early 1960’s, was to have a mix of different nozzles. At launch, the Atlas fired three rocket engines: two “boosters” optimized for sea level operation, and one “sustainer” optimized for higher altitude operation. As the rocket climbed out of the atmosphere, the “booster” rockets become less efficient, so they were discarded. The single “sustainer” rocket continued to fire as the rocket reached space.

Another approach is to have “skirts” that deploy around the nozzle as the ambient pressure drops, thus increasing the area ratio. This scheme has not been implemented on any flight vehicle. Deployable nozzle skirts *have* been used in order to make the large expansion ratio nozzles desired in space fit into the payload fairing of launch vehicles, but these are deployed before the rocket is fired.



Nozzle skirts used to match ambient pressure.

An innovative solution is called the “aerospike” nozzle. This nozzle discards the physical nozzle altogether and uses aerodynamic streamlines around the flow from the combustion chamber to create a “virtual nozzle.” Since the pressure on the streamlines must match the ambient pressure, the “nozzle” will expand so as to always match the ambient pressure as the rocket gains altitude. Aerospike nozzles were ground tested in the 1960’s and were shown to maintain near-optimal performance over the range of back pressures from 1 atm to vacuum. In the early 1970’s, the design of the future Space Shuttle even featured a rocket engine with aerospike nozzle! The lack of flight-test data has prevented their use, however. The NASA-funded X-33 program was specifically intended to flight test a linear aerospike nozzle, but has now been cancelled. Thus, the aerospike remains a promising but unused technology.

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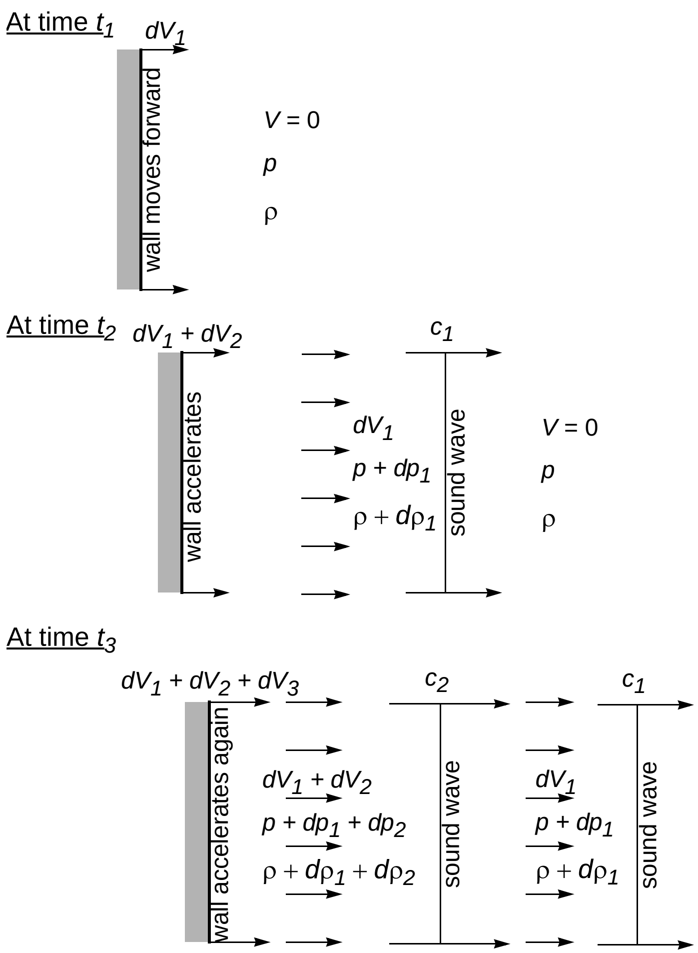
*Note the Tsiolkovski Rocket Equation: $\Delta V = I_{sp}\ln\left(\frac{m_{initial}}{m_{final}}\right)$, which shows that the payload capacity of a rocket has an *exponential* dependence on the $I_{sp}$.

# 7. Shock Waves

## 7.1 Introduction

We now turn to a new type of flow: Normal Shock Waves. To consider how normal shock waves arise, return to the picture of how a sound wave was generated in Section 4.1. Recall, we considered a wall that pushed into a compressible medium with speed $dV$, generating a disturbance that propagated outward at speed $c$. Note that the wall continues to move outward at speed $dV$, generating a region of uniform flow between the wall and the sound wave with pressure and density $p + dp$ and $\rho + d\rho$. As we showed in Section 4.1, if the wave is a compressive wave (i.e., the wall moved *into* the medium) then $dV$, $dp$, and $d\rho$ are all positive.

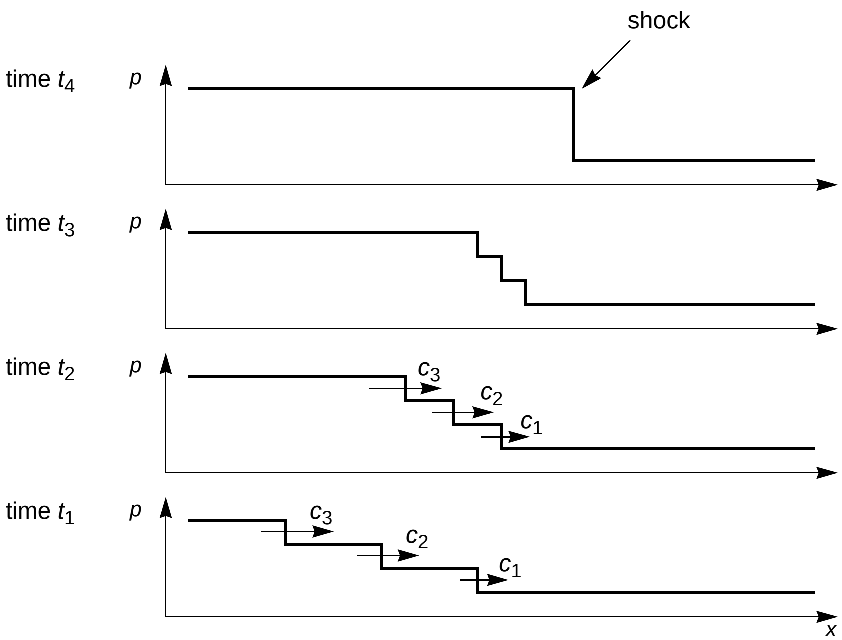
We are now going to examine what happens when a series of such waves are generated: the first one at time $t_1$, results in a increase in velocity $dV_1$, and propagates outward at speed $c_1$. A second disturbance $c_2$, generated at time $t_2$, is a result of the wall being accelerated again to a new velocity $dV_1 + dV_2$. At time $t_3$, the wall accelerates again to an even greater velocity, $dV_1 + dV_2 + dV_3$ generating a third disturbance $c_3$. Consider the propagation speeds of the three waves. Since wave $c_2$ is propagating into a medium that was already compressed by wave $c_1$, it is propagating into gas that is slightly warmer. The gas is made slightly warmer because, as discussed in Chapter 4, sound is an isentropic process, and for an isentropic compression the temperature will increase. Thus, $c_3 > c_2 > c_1$, or in other words, the sound speed tends to increase when we generate a series of compression waves. Also, note that wave $c_2$ is "riding" on top of the fluid that was already put into motion at velocity $dV_1$ by wave $c_1$. Thus, while wave $c_2$ is moving relative to the medium through which is propagates at velocity $c_2$, relative to a fixed observed, it moves at speed $c_2 + dV_1$. Likewise, wave $c_3$ moves relative to a fixed observer at velocity $c_3 + dV_1 + dV_2$. Thus, the fact that each successive wave has a higher speed of sound and is propagating through gas that has already been set into motion by the prior sound wave means that the absolute



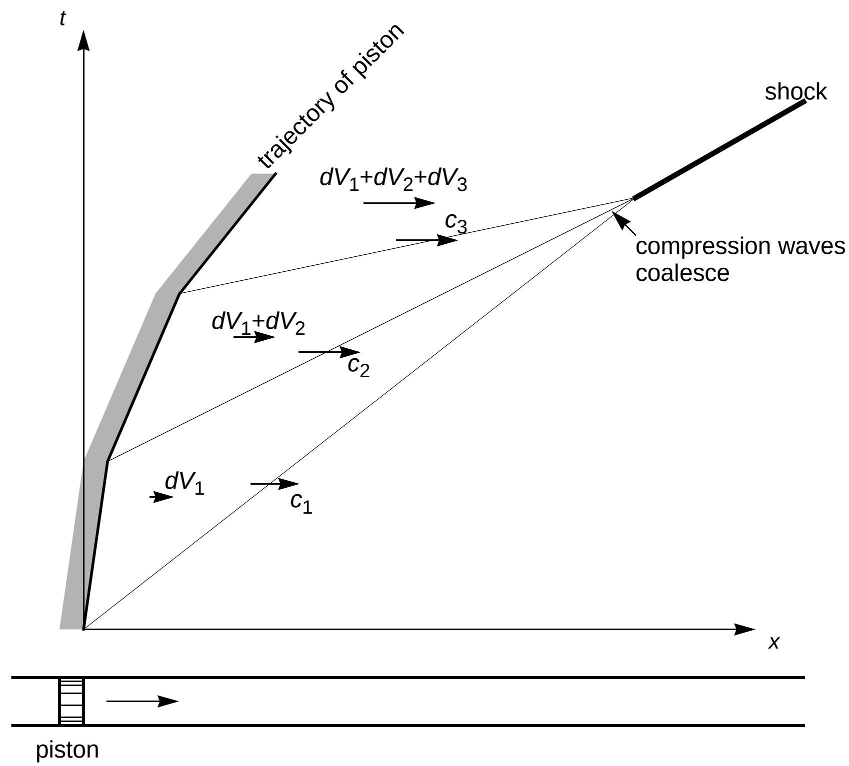

velocity of the later compression waves will always be greater than the earlier compression waves.

Because their velocity in an absolute reference frame is always increasing, the later waves in a "train" of compression waves like this will always tend to catch up with the earlier waves. The faster waves do not pass through the slower waves (if they did, the initially slower waves would then be trailing, and would become the faster waves!). Rather, the fact that the waves in the rear catch up with the waves in the front results in the waves *merging* or *coalescing* into a single wave. When enough compression waves have coalesced together, the change in properties across the wave are no longer infinitesimally small, and the jump in pressure, density, velocity, etc. becomes *finite*. We can no longer consider the wave a sound wave. We call these finite compression waves *shock waves*. We can create a shock wave directly by instantaneously accelerating the wall in the exercise above to a finite velocity $V_1$, or we can do it by the series of weak compressions shown above.

We can view this process by examine the pressure profile at various times (note that the density and temperature profiles would look qualitatively similar):




Another way to view this process is in a space-time (or *x*-*t*) diagram. Here, the *x*-axis represents the dimension along the channel, and the *y*-axis is time. Consider a piston being accelerated into a channel with initially quiescent gas:




Note that in an *x*-*t* diagram, the velocity is inversely proportional to its slope in the *x*-*t* plane. Thus, the trajectory of the piston starts off nearly vertical when its velocity is very small.*

*If you have trouble understanding an *x*-*t* diagram, make a slit in a piece of paper and align the slit with *x*-axis on the figure. Holding the slit fixed, move the figure downward, and you will see an animation of the piston and the waves moving in the channel.

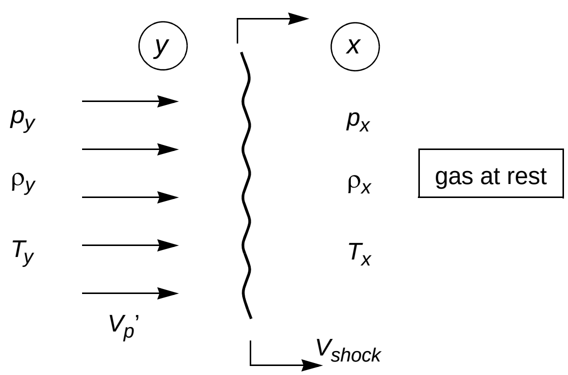
Again, we could have created the shock wave "instantaneously" by impulsively setting the piston in motion at a finite velocity. In this case, the shock wave would have been generated directly at the piston face.

Note that a series of rarefaction waves do *not* tend to catch up but rather spread out over time. The resulting rarefaction or expansion "fan" will be treated in Chapters 14-16.

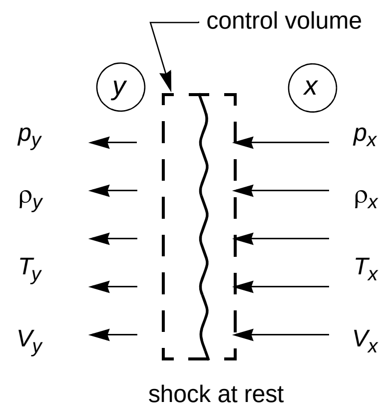
We have seen that a train of compression waves quickly coalesce into a shock wave that is extremely thin. How thin? Typically, a shock wave in gas is of the order only a few mean free paths between molecular collisions, which are typically less than microns ($< 10^{-6}$ m) apart! This is an extremely thin region that corresponds to a compression time of nanoseconds ($10^{-9}$ s) as viewed from the reference frame of the medium being processed by the shock wave. For most engineering applications, this can be considered instantaneous. Of all phenomena in Nature, the shock wave best represents the mathematical concept of *discontinuity.*

## 7.2 Normal Shock Relations

We can now consider the normal shock wave as a type of flow. (Note: the term "normal" refers to a shock that is perpendicular to the flow; we will discuss oblique shock waves in Chapter 12.) Once the shock has been formed (either by merging of compression waves or by direct impulse acceleration of the flow), the shock can be viewed as:




The conditions upstream of the shock (i.e., gas that has not yet been "processed" by the shock) will be denoted with an $x$ subscript. Conditions downstream of the shock (i.e., gas that has been shocked) will be denoted with $y$. This picture is not *steady*, since the shock is moving across the domain. We have denoted the velocity of the shock gas as $V_p$', where the prime (') is to remind us that this is the particle velocity in the unsteady reference frame (the "$p$" subscript can be either for particle velocity or piston, since this is also the velocity of piston that is necessary to create and support the shock wave). Thus, we cannot apply the tools we derived in Chapter 3 for steady, one-dimensional flow. However, we can make the shock wave steady by transforming into the shock wave's frame of reference. This is similar to what we did to make the sound wave steady in Chapter 4. This "Galilean" transformation is accomplished by adding a velocity $V_{\text{shock}}$ toward the left to everything in the picture.




Now the shock wave is fixed, and flow approaches from the right at velocity $V_x$, which is equal in magnitude to the original $V_{shock}$, but opposite in direction. The flow downstream of the shock flows away from the shock at speed $V_y$, which is equal in magnitude to $V_{shock} - V_p$'. For the most part in this course, we will be examining *steady* shock waves, so we can take the picture above (with a fixed shock) as a given. We will return to consider unsteady shocks in Chapter 8.

The control volume drawn around the shock can be taken as one-dimensional; there is no variation in properties over the flow entering or leaving the control volume. Because the shock wave is so thin, the control volume around the shock can be made very thin. This has several important consequences. For one, the fact that the control volume is thin means that the flow can be made effectively steady, even if the shock velocity or properties change with time. Consider, for example, a shock that may be decelerating with time; perhaps it loses 10% of its velocity every second. Recall that the shock wave is very thin, so the control volume can be made very thin as well (say, $10^{-6}$ m). Thus, the residence time of a fluid element as it passes through the control volume is very short ($10^{-9}$ seconds). On this time scale, the shock wave velocity is almost perfectly constant; in fact, we would have to wait many millions of these characteristic time scales in order to see any noticeable change at all in the shock velocity! Thus, the flow can be treated as steady, even though in our sense of time, the shock may be rapidly decelerating as fast as a car that has slammed on the brakes!

Similarly, the control volume can be taken small enough so that there is no time for heat transfer or momentum exchange (friction) with the surroundings, such as a wall the shock is propagating along. Thus, the shock can be treated as adiabatic and without frictional interaction with the surroundings. The thinness of the shock also permits a curved shock to be treated as a normal, one-dimensional shock. Again, we use the fact that, even a highly curved or distorted shock is normal and one-dimensional on the scale of the shock wave thickness.

The fact that the thinness of a shock permits it to accurately be treated as steady and one-dimensional means that the analysis that follows will be applicable to a wide spectrum of shock wave phenomena. In fact, these relations can be used as *exact solutions* to the shock wave problem, within the assumption of a calorically perfect gas.* A more general treatment of shock waves which is valid for any equation of state is given in Section 7.5.

Now, for the control volume show above, we have:

Continuity: $\rho \boldsymbol{V} A = \text{constant}$

Momentum: $F_{xwall} = A_y[p_y + \rho_y V_y^2] - A_x[p_x + \rho_x V_x^2]$

Energy: $\left(h + \dfrac{V^2}{2}\right) = \text{constant}$

Note that we are using the control volume (integral) form of the conservation laws. It may be tempting to use the differential form, since we are examining such a thin layer of flow across the shock. But we cannot use the differential form of the conservation laws, because the property change across the shock are discontinuous, and the derivatives of fluid properties are not defined at discontinuities. Thus, we have no choice but to apply the integral form of the conservation laws.

What we would like is to re-form these tools into something that we can use to solve the flow through a normal shock wave. Similar to isentropic flow, where given the flow conditions in one part of a channel, we could solve for the flow in other part, we would like to be able to solve for the flow leaving a normal shock if we are given the state of flow entering the shock.

If we are given $p_x$, $\rho_x$, $T_x$ and $V_x$, we would like to solve for $p_y$, $\rho_y$, $T_y$ and $V_y$. Taking inventory: we have four unknowns, and four equations to satisfy (energy, momentum, energy, and the equation of state: $p = \rho RT$). Note that the energy equation introduces another variable: $h$, but this is directly related to $T$ by the caloric equation of state: $h = c_p T$. Thus, we have a well-posed problem, and in principle we can solve the conservation equations, as given here, to find the conditions leaving the shock. As with isentropic flow, however, it becomes much more convenient to express the relations across shock wave flow in terms of Mach number.

The continuity equation can be simplified because the area entering the shock equals the area leaving the shock: $A_x = A_y$. This is another consequence of the shock being very thin, permitting the control volume to be clustered near the shock such that no area divergence can occur. Continuity thus becomes:

$$\frac{\rho_y}{\rho_x} = \frac{V_x}{V_y}$$

*An example of the confidence we have in the shock relations derived here is the fact that these relations are used to *calibrate* pressure transducers and other devices used to measure shock waves and high speed fluid dynamic phenomena.

Considering the energy equation applied to the control volume enclosing the shock:

$$\left(h_x + \frac{V_x^2}{2}\right) = h_{ox} = h_{oy} = \left(h_y + \frac{V_y^2}{2}\right)$$

Since we are dealing with a calorically perfect gas, $h = c_p T$:

$$c_p T_{ox} = c_p T_{oy}\text{, or } T_{ox} = T_{oy}$$

We can express $T_o$ in terms of the static temperature and Mach number using $\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}M^2$. Thus:

$$\frac{T_y}{T_x} = \frac{1 + \frac{\gamma - 1}{2}M_x^2}{1 + \frac{\gamma - 1}{2}M_y^2}$$

We now have the ratio of static temperatures across a shock wave in terms of the Mach numbers upstream ($M_x$) and downstream ($M_y$) of the shock. Using the ideal gas law: $p = \rho RT$, we can express the pressure ratio as: $\frac{p_y}{p_x} = \frac{\rho_y T_y}{\rho_x T_x}$.

With the continuity relation, we can replace the density ratio $\frac{\rho_y}{\rho_x}$ with $\frac{V_x}{V_y}$.

Using $V = Mc$, the pressure ratio becomes: $\frac{p_y}{p_x} = \frac{V_x T_y}{V_y T_x} = \frac{T_y M_x c_x}{T_x M_y c_y} = \frac{T_y M_x}{T_x M_y}\sqrt{\frac{T_x}{T_y}}$.

Since we already have an expression for $\frac{T_y}{T_x}$, we can express $\frac{p_y}{p_x}$ as:

$$\frac{p_y}{p_x} = \frac{M_x}{M_y}\sqrt{\frac{1 + \frac{\gamma - 1}{2}M_x^2}{1 + \frac{\gamma - 1}{2}M_y^2}}$$

The density ratio is:

$$\frac{\rho_y}{\rho_x} = \frac{p_y T_x}{p_x T_y} = \frac{M_x}{M_y}\sqrt{\frac{1 + \frac{\gamma - 1}{2}M_y^2}{1 + \frac{\gamma - 1}{2}M_x^2}}$$

These relations for temperature, pressure, and density ratio are valid for any constant-area flow without heat interaction.

We can also develop a relation for the ratio of stagnation pressure across the normal shock:

$$\frac{p_{oy}}{p_{ox}} = \frac{\frac{p_{oy}}{p_y}}{\frac{p_{ox}}{p_x}}\left(\frac{p_y}{p_x}\right)$$

We already have a relation that relates static pressure to stagnation pressure at a given point in the flow: $\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma - 1}}$. We can use this relation for the $x$ and $y$ conditions, provided we use our idealized "stagnation probe" upstream of the shock to isentropically decelerate the flow top $p_{ox}$, and downstream of the shock to decelerate the flow to $p_{oy}$. This, along with our relation for $\frac{p_y}{p_x}$, gives:

$$\frac{p_{oy}}{p_{ox}} = \frac{M_x}{M_y}\frac{\left(1 + \frac{\gamma - 1}{2}M_y^2\right)^{\frac{\gamma + 1}{2(\gamma - 1)}}}{\left(1 + \frac{\gamma - 1}{2}M_x^2\right)^{\frac{\gamma + 1}{2(\gamma - 1)}}}$$

Since a shock wave is adiabatic, $T_{ox} = T_{oy}$. In general, however, across a shock, stagnation pressure is *not* constant: $p_{ox} \neq p_{oy}$, which we will shortly demonstrate is related to the fact that a shock wave is not isentropic.

Note that these relations are not exactly what we are after, since they still have an unknown: $M_y$. But, we still have yet to invoke the momentum equation:

$$F_{xwall} = A_y[p_y + \rho_y V_y^2] - A_x[p_x + \rho_x V_x^2]$$

Again, because the control volume is so thin, there is no force interaction with the wall and $A_x = A_y$. Thus:

$$p_y + \rho_y V_y^2 = p_x + \rho_x V_x^2$$

Using the substitution $\rho V^2 = \gamma p M^2$:

$$p_y(1 + \gamma M_y^2) = p_x(1 + \gamma M_x^2)$$

$$\frac{p_y}{p_x} = \frac{1 + \gamma M_x^2}{1 + \gamma M_y^2}$$

Note that this relation for pressure is valid for any steady, constant area flow without wall force. This gives us a *second* relation for pressure ratio $\frac{p_y}{p_x}$, but both of these relations must hold for the normal shock (since it must satisfy both the momentum and energy equations). Thus, we can equate our two relations for $\frac{p_y}{p_x}$:

$$\frac{M_x}{M_y}\sqrt{\frac{1 + \frac{\gamma - 1}{2}M_x^2}{1 + \frac{\gamma - 1}{2}M_y^2}} = \frac{1 + \gamma M_x^2}{1 + \gamma M_y^2}$$

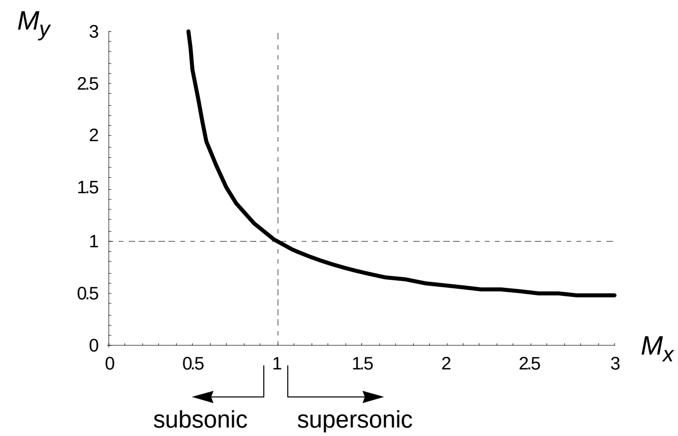
We can solve for $M_y$ by squaring both sides and expanding:

$$(\gamma - 1)(M_y^4 - M_x^4) - 2\gamma M_y^2 M_x^2 (M_y^2 - M_x^2) + 2(M_y^2 - M_x^2) = 0$$

One solution to this equation is $M_x = M_y$. For this case, we can see for the above relations for $\frac{T_y}{T_x}$ and $\frac{p_y}{p_x}$ that there will be no change in properties across the wave. Indeed, nothing happens at all in the control volume in this case: flow leaves the control volume exactly as it entered. If we are interested in the case of a finite jump in properties across the shock, then $M_x \neq M_y$. We can use the fact that $M_y^2 - M_x^2 \neq 0$ to divide through by $M_y^2 - M_x^2$ and solve for $M_y$:

$$M_y = \sqrt{\frac{2 + (\gamma - 1)M_x^2}{2\gamma M_x^2 - (\gamma - 1)}}$$

We can plot this function as a curve giving the downstream Mach number of the flow leaving the shock $M_y$ as a function of the upstream Mach number of the flow approaching the shock $M_x$.



Note that there are two branches to the solution: one with supersonic flow approaching the shock and subsonic flow leaving (the *supersonic* shock), and the other with subsonic flow approaching the shock and supersonic flow exiting (the *subsonic* shock). Note that by examining the temperature and pressure relations:

$$\frac{T_y}{T_x} = \frac{1 + \frac{\gamma - 1}{2} M_x^2}{1 + \frac{\gamma - 1}{2} M_y^2} \text{ and } \frac{p_y}{p_x} = \frac{1 + \gamma M_x^2}{1 + \gamma M_y^2}$$

we can conclude that the supersonic shock is a *compressive* shock that increases pressure and temperature (and density), while the subsonic shock is a *rarefaction* shock that lowers pressure and temperature. The case of the rarefaction shock is already suspect, because as we mentioned in considering of the "train" of sound waves, that a train of rarefaction waves would tend to spread out and a shock would not occur. The rarefaction shock is also difficult to imagine because it would be a discontinuity in pressure that travels at speeds less than the speed of sound. If you had a discontinuity in pressure, would you not expect this to result in pressure waves that would propagate at speeds at least equal to the speed of sound?

We will definitely rule out the case of the subsonic, rarefaction shock in a minute, but it is worth pointing out that this solution is simply the flow of the supersonic flow run backwards. The function $M_y(M_x)$ is symmetric, meaning that if we take the output $M_y$ and evaluate the function at this value, it will generate the original value of $M_x$. This is because the two solutions are really the *same* flow, just run the opposite direction. It is as if we made a movie of the flow through the supersonic shock, and then played it in reverse: subsonic flow approaches the shock, and supersonic flow streams away.

There are many instances in physics and engineering in which we can make a movie of a phenomenon and play it backwards and observe a similarly feasible phenomenon. The motion of the planets around the sun, the motion of a pendulum, or gentle waves on the surface of a lake, can all be played forward and backward and appear equally valid. But other processes, such a dropping and breaking a coffee cup, the motion of a waterfall, or air being drawn into the intake of an engine and combustion products coming out the exhaust pipe, clearly become ridiculous when played in reverse. In order to provide a systematic way to determine if a process is directional or irreversible, the concept of *entropy* is introduced.

Recall from thermodynamics that entropy is defined by

$$Tds = du + pdv$$

$$Tds = dh - vdp$$

For a calorically perfect gas, the second expression becomes

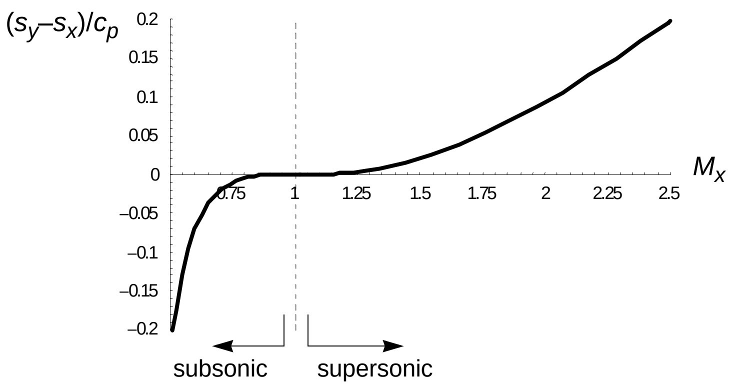
$$Tds = c_p dT - vdp$$

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

This expression can be integrated in closed form:

$$\Delta s = s_y - s_x = c_p \ln\frac{T_y}{T_x} - R\ln\frac{p_y}{p_x} = c_p \ln\frac{\frac{T_y}{T_x}}{\left(\frac{p_y}{p_x}\right)^{\frac{\gamma-1}{\gamma}}}$$

Since we have expressions for $\frac{T_y}{T_x}$ and $\frac{p_y}{p_x}$ as functions of $M_x$ and $M_y$, and we can now express $M_y$ as a function of $M_x$, it is possible to plot then entropy change across a shock as a function of the shock Mach number $M_x$.




Note that for a shock Mach number of less than $M_x = 1$, the entropy *decreases* across the shock, which is a violation of the 2nd Law of Thermodynamics for an adiabatic process. Thus, we can definitively rule out the subsonic, expansion shock wave case. *Only supersonic, compressive shock waves are observed in Nature*.* Note from the plot above that shock waves with strength greater than Mach 1 result in an increase in entropy. Thus, shock waves are *not* isentropic and in general are very strong entropy generating mechanisms.

Now that we have $M_y = \text{func}(M_x)$, we can re-write our expressions for $\frac{T_y}{T_x}$, $\frac{p_y}{p_x}$, and $\frac{\rho_y}{\rho_x}$ as functions of $M_x$ alone:

$$\frac{T_y}{T_x} = \frac{[2\gamma M_x^2 - (\gamma - 1)][2 + (\gamma - 1)M_x^2]}{(\gamma + 1)^2 M_x^2}$$

$$\frac{p_y}{p_x} = \frac{2\gamma M_x^2 - (\gamma - 1)}{\gamma + 1}$$

$$\frac{\rho_y}{\rho_x} = \frac{V_x}{V_y} = \frac{(\gamma + 1)M_x^2}{2 + (\gamma - 1)M_x^2}$$

$$\frac{p_{oy}}{p_{ox}} = \frac{A_x^*}{A_y^*} = \left[\frac{2\gamma M_x^2 - (\gamma - 1)}{\gamma + 1}\right]^{\frac{-1}{\gamma - 1}}\left[\frac{(\gamma + 1)M_x^2}{2 + (\gamma - 1)M_x^2}\right]^{\frac{\gamma}{\gamma - 1}}$$

The numerical value of these relations are tabulated in Appendix Table A3 for $\gamma = 1.4$.

The fact that $\frac{p_{oy}}{p_{ox}} = \frac{A_x^*}{A_y^*}$ derives from the fact that the mass flow rate at any point in a flow can be expressed as a function of $A^*$, $p_o$, and $T_o$, as was derived in Section 5.4:

$$\frac{\dot{m}}{A_x^*} = \sqrt{\frac{\gamma}{R}}\frac{p_{ox}}{\sqrt{T_{ox}}}\sqrt{\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma + 1}{\gamma - 1}}}, \quad \frac{\dot{m}}{A_y^*} = \sqrt{\frac{\gamma}{R}}\frac{p_{oy}}{\sqrt{T_{oy}}}\sqrt{\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma + 1}{\gamma - 1}}}$$

The mass flow rate entering the shock equals the mass flow rate exiting the shock, from continuity: $\dot{m}_x = \dot{m}_y$. Since stagnation temperature is constant across a shock ($T_{ox} = T_{oy}$), the change in stagnation pressure is inversely related to the change in sonic area:

*...for an ideal gas. Whether an expansion shock can be observed in other media is still an open question.

$$\frac{p_{oy}}{p_{ox}} = \frac{A_x^*}{A_y^*}$$

This can also be written as:

$$p_{ox}A_x^* = p_{oy}A_y^*$$

Thus, while $p_o$ and $A^*$ are *not* constant across a shock, their *product is constant.* This relation becomes useful in problem solving.

The ratio of stagnation pressure across a shock $\frac{p_{oy}}{p_{ox}}$ can also be related to the production of entropy as follows. If we compute the change in stagnation entropy, we can use the same relation we derived above for $\Delta s$, because it is valid between any two points in a flow:

$$\Delta s_o = s_{oy} - s_{ox} = c_p \ln\frac{T_{oy}}{T_{ox}} - R\ln\frac{p_{oy}}{p_{ox}}$$

Since a shock wave is adiabatic ($T_{ox} = T_{oy}$), the first term vanishes. Also, recall from the definition of stagnation conditions, the stagnation conditions of a flow are the properties if the flow is decelerated isentropically to zero velocity. Thus, $s_{ox} = s_x$ and $s_{oy} = s_y$. The change in entropy is then

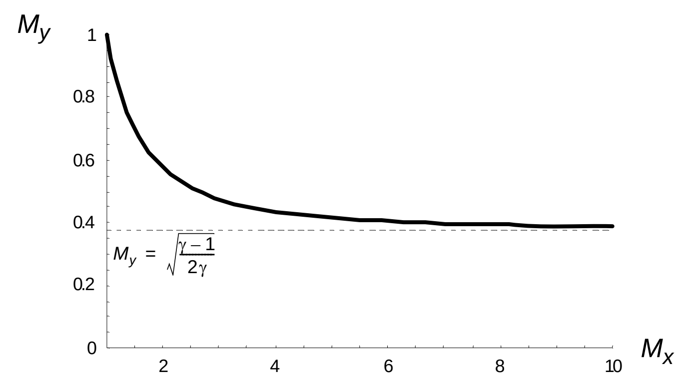
$$\Delta s = s_y - s_x = -R\ln\frac{p_{oy}}{p_{ox}}$$

Thus, the increase in entropy is directly related to the ratio of stagnation pressure across the shock. Since entropy must increase ($s_y - s_x > 0$), the ratio of stagnation pressure must be less than one: $\frac{p_{oy}}{p_{ox}} < 1$. In other words, a increase in entropy is associated with a decrease in stagnation pressure.

The link between entropy increase and stagnation pressure loss is an important one. Entropy, while crucial to the development of thermodynamics, is a difficult concept to grasp or visualized. You cannot buy an "entropimeter" from a supplier of scientific instruments. But engineers can and do measure stagnation pressure, using a pitot gauge, as discussed in Section 9.1. It is very common for engineers to discuss the stagnation pressure loss (i.e., $p_{ox} - p_{oy}$) that is experienced as flow passes through a device (an inlet on an engine, for example). When an engineer says a given device produces "a lot of total pressure loss," they are actually referring to an excessive production of entropy, which directly effects the overall device efficiency. We can now see that a decrease in stagnation pressure, and the associated increase in sonic area, is directly related to an increase in entropy.

## 7.3 Strong Shock Relations

It is of interest to examine the conditions across a shock as the shock becomes very strong. The strength of a shock wave is usually quantified by the Mach number of the shock $M_x$ or the pressure ratio $\frac{p_y}{p_x}$. In the limit of a very strong shock ($M_x >> 1$, $\frac{p_y}{p_x} >> 1$), we can see from the plot of $M_y$ vs. $M_x$ that the subsonic downstream Mach number asymtototes to a value:



$$\lim_{M_x \to \infty} M_y = \sqrt{\frac{\gamma - 1}{2\gamma}} = 0.378 \text{ (for } \gamma = 1.4\text{)}$$

Thus, while the upstream Mach number $M_x$ can increase without bound, the downstream Mach number asymptotes to a finite value.

If we examine the other relations:

$$\lim_{M_x \to \infty} \frac{T_y}{T_x} = \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2} M_x^2$$

$$\lim_{M_x \to \infty} \frac{p_y}{p_x} = \frac{2\gamma}{\gamma + 1} M_x^2$$

$$\lim_{M_x \to \infty} \frac{\rho_y}{\rho_x} = \lim_{M_x \to \infty} \frac{V_x}{V_y} = \frac{\gamma + 1}{\gamma - 1} = 6 \text{ (for } \gamma = 1.4\text{)}$$

Note that, for strong shocks, temperature and pressure increase with the square of the shock Mach number, apparently without bound. In reality, when the shock Mach number is sufficiently strong (typically, $M_x > 5$), most gases become hot enough to excite other degrees of freedom, such as molecular vibration. As the shock strength continues to increase, the gas chemically disassociates, and then ionizes. The gas in these cases can no longer be considered calorically perfect, and the value of $\gamma$ is no longer constant. These effects* tend to limit the temperature of the gas; the energy of the shock goes into dissociating and ionizing the gas rather than heating it. This feature distinguishes *hypersonic* flow from supersonic flow. In this course, we will mostly focus on shock Mach numbers less than the hypersonic regime ($M_x < 5$).

*These high-temperature effects are sometimes referred to as "real gas" effects. This is actually a poor choice of words, since "real gas" refers to conditions when the ideal gas equation of state ($p = \rho RT$) breaks down, as typically occurs at high pressures. The high-temperature effects are better referred to as "non-perfect gas" effects, meaning the gas is no longer calorically perfect and $c_p$ and $c_v$ become functions of temperature. In the hypersonic regime, the ideal gas equation of state ($p = \rho RT$) is usually still valid.

The density ratio does not increase without bound, but asymtotes to a finite value of $\frac{\rho_y}{\rho_x} = \frac{\gamma + 1}{\gamma - 1}$. Thus, a shock wave cannot create fluid of arbitrarily high density. The same is true of the velocity ratio across a shock, which is the reciprocal of the density ratio.

## 7.4 Weak Shock Relations

In contrast to the last section, it is also interesting to examine the case of weak shock waves, where $M_x$ is not much larger than unity. It can be shown in this limit that the shock relations become:

$$\lim_{M_x \to 1} \frac{T_y}{T_x} = 1 - 2\left(\frac{\gamma - 1}{\gamma + 1}\right)(M_x^2 - 1)$$

$$\lim_{M_x \to 1} \frac{p_y}{p_x} = 1 + \frac{2\gamma}{\gamma + 1}(M_x^2 - 1)$$

$$\lim_{M_x \to 1} \frac{\rho_y}{\rho_x} = 1 + \frac{2}{\gamma + 1}(M_x^2 - 1)$$

$$\lim_{M_x \to 1} \frac{\Delta s}{R} = \frac{2\gamma}{3(\gamma + 1)^2}(M_x^2 - 1)^3$$

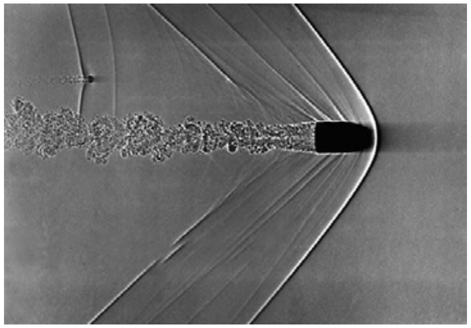
Note the last relation for the entropy increase depends on $(M_x^2 - 1)^3$, which is an extremely sensitive function as the shock Mach number increases above unity. For weak shock waves, where this term is very small ($M_x < 1.25$), the entropy increase is very small. Thus, weak shock wave are very nearly isentropic. As the shock Mach number increases, this term grows very rapidly.

## 7.5 Rankine-Hugoniot Relations

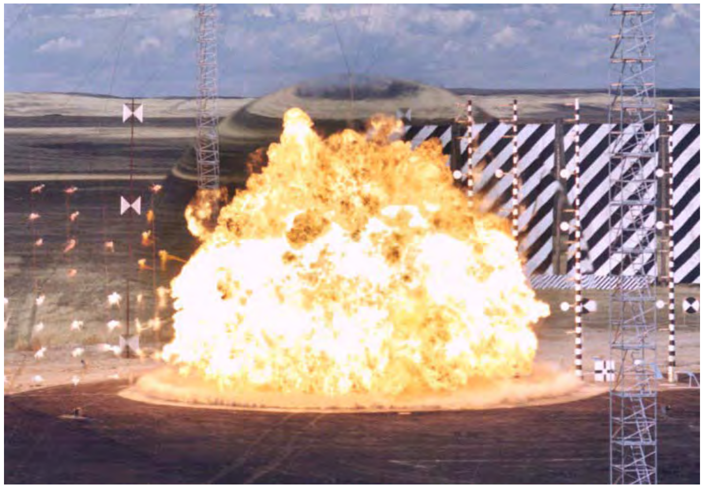
[to be added later]

## 7.5 Picture Gallery

Shock waves visible on the surface of the water, generated from the near simultaneous firing of two 16 inch guns on the battleship *Iowa*. The shell travelling down the barrel of the gun acts as a piston, pushing a shock wave ahead. When the shell leaves the barrel, the expansion of the combustion products also acts as a piston. Together, these effects create a strong shock known as the “muzzle blast” that expands spherically outward from the muzzle of the gun.



Steady bow shock around a supersonic projectile. The projectile (a bullet) drives a normal shock into the gas ahead of it. As the shock moves away from the body, it becomes an oblique shock and eventually decays to a Mach wave. Note additional oblique shocks appear as the flow interacts with the bullet surface and turbulent wake.



Shock wave generated by the detonation of a large charge of TNT. Shock wave is visible as a distortion in air, being pushed ahead by the fireball of TNT combustion products. Picture from Defence R&D Canada, Suffield, Alberta.

Picture from the Hubble Space Telescope showing a bow shock visible around the star LL Ori in the Orion Nebula (1500 light years from Earth). Gas flows from the Orion Nebula (located in the lower right of this picture) until it collides with higher velocity gas flowing from the young star LL Ori. The result is a bow shock, similar to that created by a supersonic body in air. Note another bow shock is visible around another star in the upper right corner.

Picture taken by Hubble Space Telescope of OH231.8+4.2, a dying star that is ejecting jets of gas at velocities of 500 km/s. The jets (orange in the color picture) shoot out from opposite sides of the star (not visible behind a dust cloud) and impact the surrounding media to produce a bow shock (blue in the color picture). The resulting structure is called the “Calabash Nebula” and is 1.4 light-years long and located 5000 light-years from Earth.

# 8. Flow with Shock Waves

Now that we are equipped with the normal shock relations, we can return to the problem of the converging-diverging nozzle and consider the case where a shock wave occurs in the supersonic, diverging portion of the nozzle.

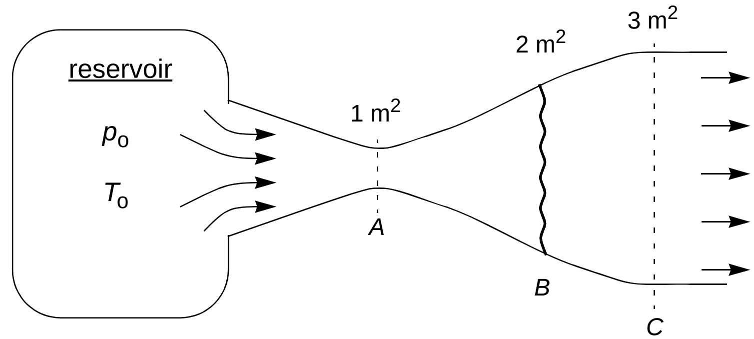
## 8.1 Solving Flows with Shock Waves

The procedure for solving flows with shock waves will be similar to the procedure used for isentropic flows. We will always begin by finding the reference states (stagnation or sonic conditions). We will use these to solve isentropic flow up to the point where the shock wave is located, and then apply the normal shock relations. We must exercise caution in dealing with the shock wave: some reference conditions ($T_o$, for example) are constant across the shock, other reference conditions ($p_o$ and $A^*$) are not. After the shock, we continue to use isentropic flow relations to solve the flow downstream of the shock, remembering to use the new reference conditions determined by the shock.

This is best illustrated with a numerical example.

### 8.1.1 Numerical Example: Converging-Diverging Nozzle with Shock in Nozzle

**Problem:** Consider a converging-diverging nozzle fed by a reservoir of quiescent air with stagnation pressure $p_o$ = 1 atm and stagnation temperature $T_o$ = 300 K. We are told the flow is isentropic, except for a single normal




shock (stationary) at location *B*. Find the pressure, temperature, and Mach number at locations *A*, *B*, and *C*. The area [m$^2$] at these locations is given in the figure.

**Solution:**

Step 1: Find the reference states. We were told that $p_o$ = 1 atm and $T_o$ = 300 K Further, since we are told of the existence of a shock in the diverging nozzle, the flow must be supersonic in the diverging nozzle, because stationary shock waves can only occur in supersonic flow. Thus, the flow must have passed through a sonic throat at location *A*. Therefore, the sonic area must be $A^*$ = 1 m$^2$.

Step 2: Solve the isentropic flow upstream of the shock wave. The conditions at location *A* are simply the sonic conditions:

$$\frac{p_A}{p_o} = \frac{p^*}{p_o} = 0.5283 \Rightarrow p_A = 0.5283 \text{ atm}$$

$$\frac{T_A}{T_o} = \frac{T^*}{T_o} = 0.8333 \Rightarrow T_A = 250 \text{ K}$$

The conditions at location *B* just upstream of the shock (denoted *Bx*) are found from the area ratio $A_{Bx}/A^*$:

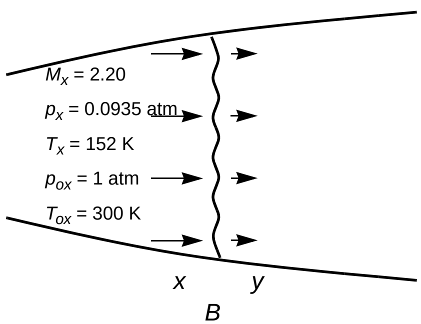
$$\frac{A_{Bx}}{A^*} = 2 \Rightarrow M_{Bx} = 2.20$$

Note that we took the supersonic solution for this area ratio because we knew the flow had to be supersonic, due to the presence of the normal shock. Knowing the Mach number, we can solve for the static pressure and temperature:

$$\left(\frac{p_{Bx}}{p_o}\right)_{M\,=\,2.2} = 0.0935 \Rightarrow p_{Bx} = 0.0935 \text{ atm}$$

$$\left(\frac{T_{Bx}}{T_o}\right)_{M\,=\,2.2} = 0.5083 \Rightarrow T_{Bx} = 152 \text{ K}$$

Step 3: Solve flow across shock wave. Since the shock is stationary, the flow entering the shock is the same as the solution for the isentropic flow at location *Bx*.




Using the working relations for a normal shock, evaluated at $M_x = 2.20$ (or, using the normal shock tables):

$$M_x = 2.20 \Rightarrow M_y = 0.547$$

$$\left(\frac{p_y}{p_x}\right)_{M\,=\,2.20} = 5.48 \Rightarrow p_y = 0.512 \text{ atm}$$

$$\left(\frac{T_y}{T_x}\right)_{M\,=\,2.20} = 1.857 \Rightarrow T_y = 282 \text{ K}$$

$$\left(\frac{p_{oy}}{p_{ox}}\right)_{M\,=\,2.20} = 0.628 \Rightarrow p_{oy} = 0.628 \text{ atm}$$

$$\frac{A_y^*}{A_x^*} = \frac{p_{ox}}{p_{oy}} = \frac{1}{0.628} = 1.59 \Rightarrow A_y^* = 1.59 \text{ m}^2$$

Note that the stagnation temperature does not change across a shock wave, so $T_{oy} = T_{ox} = 300$ K. To verify this, we can compute $T_{oy}$ from the properties downstream of the shock:

$$T_{oy} = T_y\left(1 + \frac{\gamma - 1}{2}M_y^2\right) \approx 299 \text{ K}$$

The value of $T_{oy} \approx 299$ K is within round-off error of the correct value $T_{oy} = 300$ K. We now know all of the conditions downstream of the normal shock.

Step 4: Solve isentropic flow downstream of shock. Note that the reference states have changed: we must now use $p_{oy} = 0.628$ atm and $A_y^* = 1.59 \text{ m}^2$ as the stagnation pressure and sonic area. Since we know the area at station C, we can find the Mach number from the area ratio $\frac{A_c}{A^*}$:

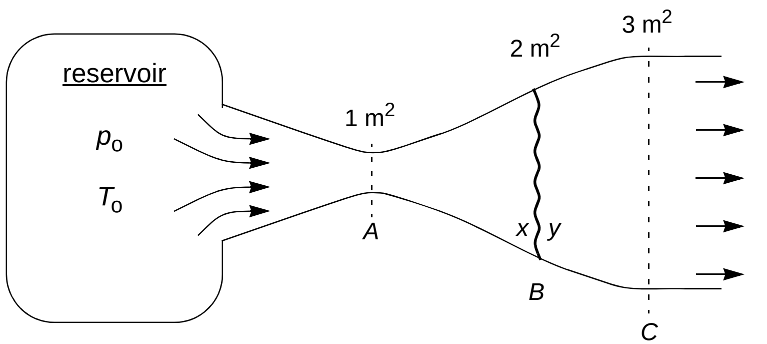
$$\frac{A_c}{A^*} = \frac{3}{1.59} = 1.88 \Rightarrow M_c = 0.33$$

Note that we use the subsonic solution, since we know the flow downstream of a normal shock is subsonic, and in a diverging area, a subsonic flow can only decelerate. Now that we know Mach number, we can solve for pressure and temperature:

$$\left(\frac{p_c}{p_o}\right)_{M\,=\,0.33} = 0.927 \Rightarrow p_c = 0.58 \text{ atm}$$

$$\left(\frac{T_c}{T_o}\right)_{M\,=\,0.33} = 0.979 \Rightarrow p_c = 294 \text{ K}$$

To conclude, we will make a table summarizing our results:




| | reservoir | *A* | *Bx* | *By* | *C* |
|---|---|---|---|---|---|
| *M* | 0 | 1 | 2.2 | 0.54 | 0.33 |
| *p* | 1 atm | 0.53 atm | 0.094 atm | 0.51 atm | 0.58 atm |
| *T* | 300 K | 250 K | 152 K | 282 K | 294 K |
| $p_o$ | 1 atm | 1 atm | 1 atm | 0.63 atm | 0.63 atm |
| $T_o$ | 300 K | 300 K | 300 K | 300 K | 300 K |
| $A^*$ | 1 m$^2$ | 1 m$^2$ | 1 m$^2$ | 1.59 m$^2$ | 1.59 m$^2$ |

Note that chamber which the nozzle discharges into must provide a back pressure $p_{\text{back}} = p_{\text{c}} = 0.58$ atm, since the exit flow is subsonic.

## 8.2 Converging-Diverging Nozzle with Variable Back Pressure

We can now return to the problem of the converging-diverging nozzle with variable back pressure. Now that we can treat the case of a normal shock appearing in the supersonic flow in the diverging nozzle section, we see that cases involving a back pressure less than $p_3$ but greater than $p_{\text{d}}$ will result in the formation of a shock wave. A shock wave is the only mechanism that can reconcile a back pressure that is too great for supersonic flow through the diverging nozzle and too small for subsonic flow through the diverging nozzle.

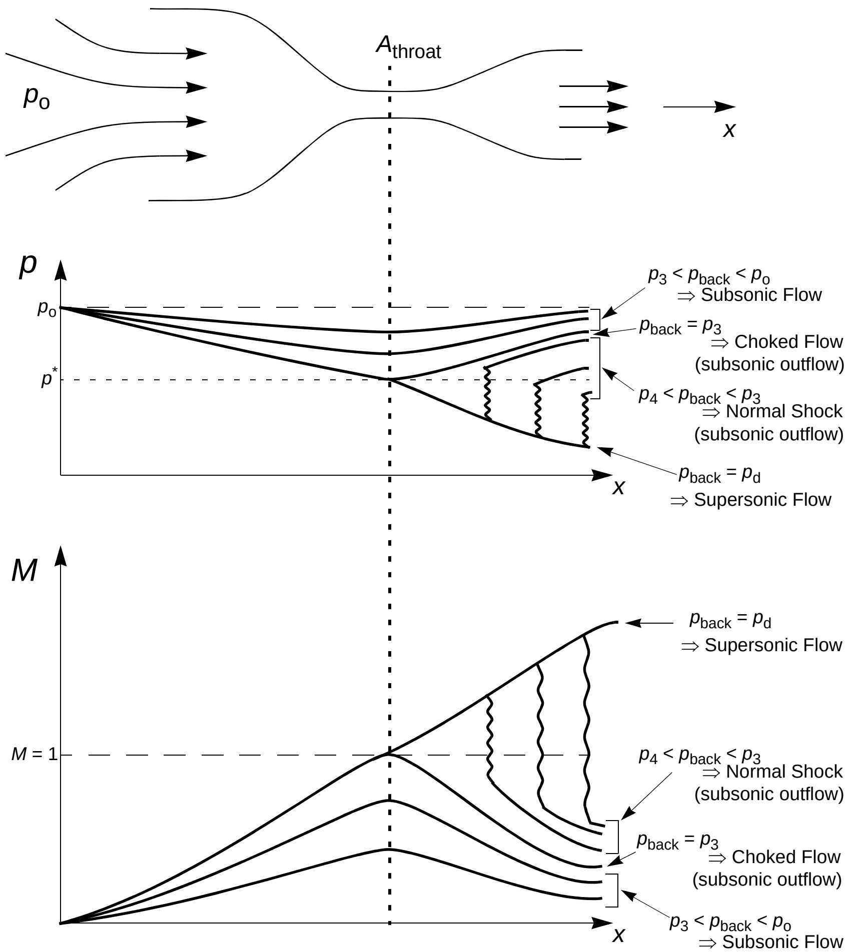
As the back pressure is decreased incrementally below the $p_3$ case (sonic at throat, but subsonic in the diverging section), a weak normal shock will appear immediately downstream of the throat. As the back pressure is decreased, the normal shock will recede further downstream and its strength will increase, as it now occurs in flow that increases in Mach number as we proceed downstream. Finally, for the case of back pressure $p_4$, the shock wave is located at the exit plane of the nozzle.

What happens for back pressures between $p_4$ and $p_{\text{d}}$? This is case that we will have to wait until we have covered oblique shock waves (Chapter 12). Suffice for now to say that the normal shock will move out of the nozzle and become an oblique shock (or system of oblique and normal shocks) that act to bring the exit pressure to the ambient pressure.

We can summarize these cases as follow:

- $p_{\text{back}} \geq p_3 \quad \Rightarrow$ Subsonic flow through diverging nozzle
- $p_3 > p_{\text{back}} > p_4 \Rightarrow$ Normal shock in diverging nozzle
- $p_4 > p_{\text{back}} > p_{\text{d}} \Rightarrow$ Oblique shocks outside diverging nozzle
- $p_{\text{back}} = p_{\text{d}} \quad \Rightarrow$ Supersonic flow through diverging nozzle

For problems in which we are given fixed reservoir conditions, a nozzle profile, and a given back pressure, it is helpful to compute the cases of $p_3$ and

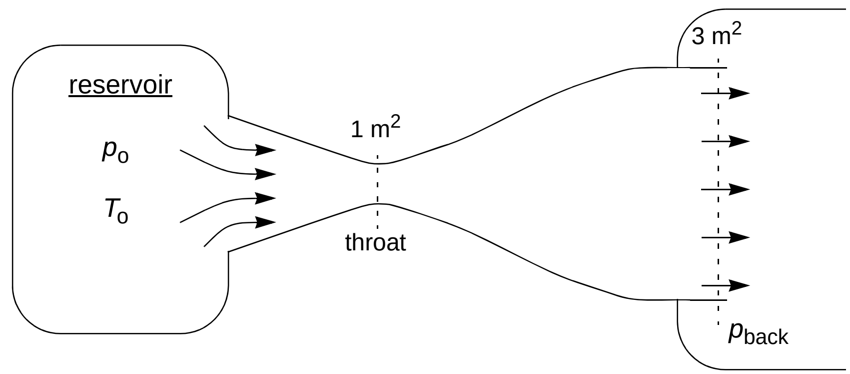



$p_4$. Once these cases are know, the actual flow that occurs for these given conditions can be quickly identified. If a normal shock is present in the nozzle, its location can be determined by procedure of trial-and-error positioning of the shock and computing the exit pressure using isentropic flow and normal shock relations. If the exit pressure does not match the back pressure, the location of the shock is adjusted and the exit pressure re-computed. This procedure is continued until the correct location of the shock is identified.

### 8.2.1 Numerical Example: Converging-Diverging Nozzle with Given Back Pressure

Let us return to the problem considered in Section 8.1.1 and, instead of specifying the shock location, we will specify the back pressure.

**Problem:** Consider a converging-diverging nozzle fed by a reservoir of quiescent air with stagnation pressure $p_o = 1$ atm and stagnation temperature $T_o = 300$ K. The nozzle discharges into a chamber with a fixed back pressure




$p_{back} = 0.7$ atm. The throat area of the nozzle is $A_{throat} = 1\ m^2$ and the exit area is $A_{exit} = 3\ m^2$. Determine if a shock wave is present. If it is, determine the area at the location of the shock.

<u>Step 1:</u> Assume sonic at throat, *subsonic* in diverging section. Since the flow is choked at the throat, then $A^* = 1\ m^2$, so:

$$\frac{A_{exit}}{A^*} = 3.00 \Rightarrow M_{exit} = 0.20$$

Knowing exit Mach number, we can find the exit pressure:

$$\left(\frac{p}{p_o}\right)_{M\,=\,0.20} = 0.97 \Rightarrow p_{exit} = 0.97 \text{ atm}$$

This exit pressure corresponds to the case $p_3$. Since the actual back pressure is less than this case, we clearly do not have subsonic flow throughout the diverging nozzle. The flow must have become supersonic at some point.

Step 2: Assume sonic at throat, *supersonic* in diverging section (i.e., the "design" case). Since the flow is choked at the throat, then $A^* = 1$ m$^2$, so:

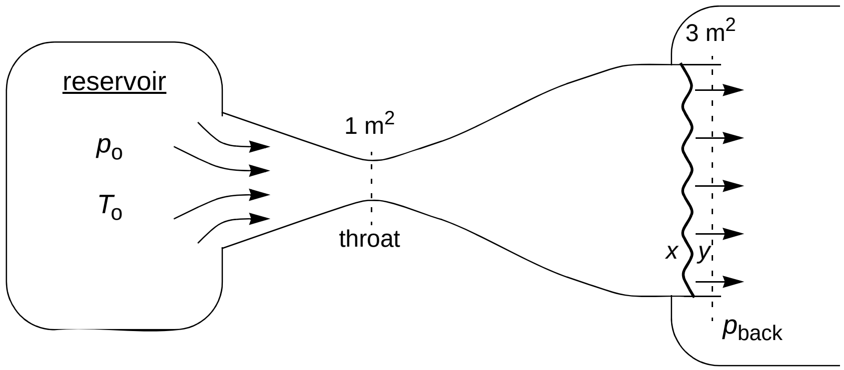
$$\frac{A_{exit}}{A^*} = 3.00 \Rightarrow M_{exit} = 2.64$$

Again, knowing exit Mach number, we can find the exit pressure:

$$\left(\frac{p}{p_o}\right)_{M\,=\,2.64} = 0.047 \Rightarrow p_{exit} = 0.047 \text{ atm}$$

This exit pressure is much less than the actual back pressure. Thus, we are not at the "design" case of $p_d$, and the pressure mismatch could result in a normal shock appearing in the nozzle.

Step 3: Assume normal shock at exit plane.




In this case, we already know the flow entering the normal shock: this is the flow that would be exiting the nozzle at "design" conditions, as we found in Step 2 above. If we apply the normal shock relations to $M_x = 2.64$ and $p_x = 0.047$ atm, we obtain:

$$M_x = 2.64 \Rightarrow M_y = 0.501$$

$$\left(\frac{p_y}{p_x}\right)_{M_c\,=\,2.64} = 7.97 \Rightarrow p_y = 0.374 \text{ atm}$$

This corresponds to case $p_4$ (shock at nozzle exit plane). This exit pressure still does not match the back pressure, but we are definitely closing in: the fact that the back pressure is between $p_3$ and $p_4$ ($p_3 > p_{back} > p_4$) informs us that there is definitely a normal shock in the diverging section. Now, it is just a matter of locating it.

Step 4: Trial location of normal shock. We can guess a location of the normal shock. A convenient first guess may be at the halfway point in the nozzle area profile, where $A = 2$ m$^2$.* We can now apply isentropic flow up to the normal shock, normal shock relations across the shock, and then isentropic flow downstream of the shock. We have already done this calculation, however, in Section 8.1.1. Recall the result we found was: $p_{exit} = 0.58$ atm. This still does not match the back pressure, but we are getting closer.

Step 5: Iterate on location of normal shock. Since positioning the shock at the midpoint resulted in too low an exit pressure, we must move the shock upstream. This will result in a weaker shock, giving a smaller total pressure loss and a larger exit pressure. So, we will attempt a guess at $A_{shock} = 1.5$ m$^2$. The supersonic Mach number at that location is:

$$\frac{A}{A^*} = \frac{1.5}{1} = 1.5 \Rightarrow M_x = 1.85$$

---

*Note that you may be tempted to try guessing the shock at the nozzle throat in order to bound its location. A shock at the nozzle throat, however, would be a sound wave, since the flow is sonic there. We have already computed this case (sonic at throat, subsonic in diverging nozzle) in Step 1.

For a Mach 1.85 shock, the change in stagnation pressure is:

$$\frac{p_{oy}}{p_{ox}} = \frac{A_x^*}{A_y^*} = 0.790 \Rightarrow p_{oy} = 0.790 \text{ atm}$$

We can find the exit Mach number from:

$$\frac{A_{exit}}{A_y^*} = \frac{A_{exit}}{A_x^*}\frac{A_x^*}{A_y^*} = \frac{3}{1} \cdot 0.790 = 2.37 \Rightarrow M_{exit} = 0.25$$

Knowing $M_{\text{exit}}$ and $p_{\text{oy}}$, we can find the exit pressure:

$$\left(\frac{p}{p_o}\right)_{M = 0.25} = 0.957 \Rightarrow 0.76 \text{ atm}$$

This is slightly greater than the actual back pressure. We can iterate again, but with the solutions for the shock at 1.5 m$^2$ and 2.0 m$^2$, we can interpolate:

| $A_{\text{shock}}$ | $p_{\text{exit}}$ |
|---|---|
| 2.0 m$^2$ | 0.58 atm |
| 1.5 m$^2$ | 0.76 atm |

To estimate the actual area at the shock location:

$$A_{shock} \approx 1.5 + \left(\frac{0.7 - 0.76}{0.58 - 0.76}\right)(2.0 - 1.5) \approx 1.67$$

We can now perform another calculation using $A_{\text{shock}}$ = 1.67 m$^2$ and continue iterating, but this value is sufficient for an engineering calculation.

## 8.3 Real Flows in Supersonic Nozzles

Shock systems still respond to changes in back pressure in the same way that a single, normal shock does. Thus, the single normal shock is still a useful idealization for shock waves in converging diverging nozzles.

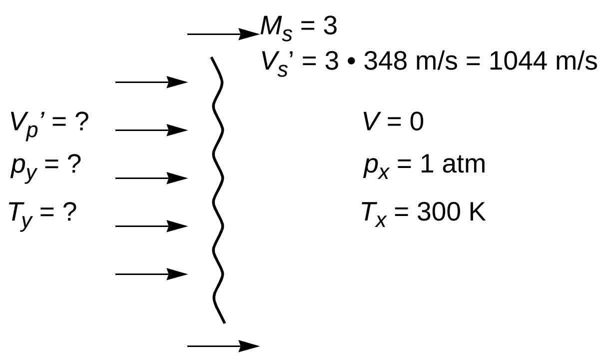
# 8.4 Unsteady Shock Waves

Thus far, we have been examining flows in which the shock wave is fixed at given location in the nozzle. Recall in our introduction to shock waves (Section 7.1), however, that we discussed the generation of a moving shock wave by the motion of a piston. We made this shock wave *steady* by transforming into its reference frame. In this section, we will transform back into the *unsteady* reference frame where the shock wave is moving. We will use the same normal shock relations we developed in Chapter 7, but we now must exercise caution that these relations are *only* applied in the *shock-fixed, steady reference frame.*

To illustrate this point, we will consider a numerical example:

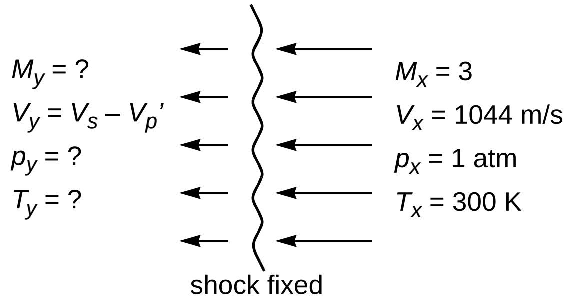
## 8.4.1 Numerical Example: Unsteady Shock Wave

**Problem:** Consider a shock wave propagating at Mach 3 into quiescent (i.e., at rest) air at standard temperature and pressure ($p$ = 1 atm, $T$ = 300 K). Find the pressure, temperature, and particle velocity of the air after passage of the shock wave.




**Solution:** In the figure above, we have shown the particle velocity to the right (in the same direction as the shock wave). Recall from Chapter 7 that a prime superscript (’) denotes a velocity in the unsteady or “lab-fixed” reference frame. Also recall from Section 7.1 that we can imagine a piston behind the shock, maintaining the gas particle velocity at $V_p$’. The particle velocity then must be to the right (otherwise, we would have a rarefaction wave rather than a shock wave).

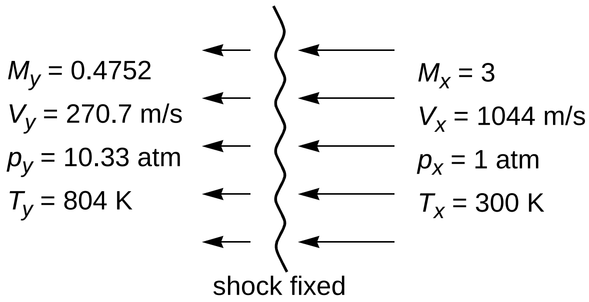
If we travel along with a shock wave propagating into air at Mach 3, then the shock wave will be at rest and we will see the air approaching at Mach 3. In order to make this transformation of reference frame, you need to add the negative velocity of the shock ($V_s$ = 1044 m/s) to every velocity in the picture above to obtain:



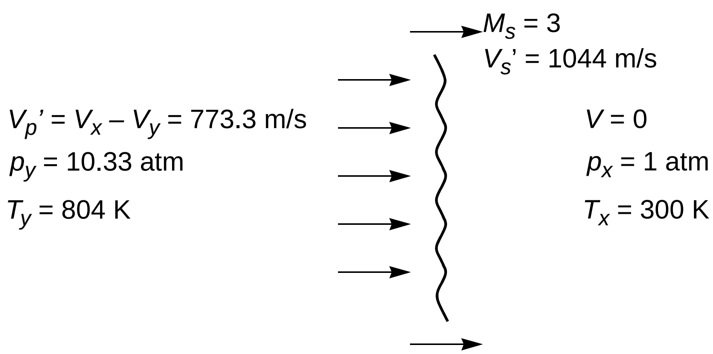

Note that static properties ($p_x$, $T_x$, $p_y$, $T_y$) do not change as we transform reference frame. Now that we have made the shock steady, we can apply the normal shock relations (or use the normal shock table) to find:

$$M_x = 3.0 \Rightarrow M_y = 0.4752 \,, \frac{p_y}{p_x} = 10.33 \,, \frac{\rho_y}{\rho_x} = 3.857 \,, \frac{T_y}{T_x} = 2.679$$

Thus, we can find $p_y$ = 10.33 atm, $T_y$ = 804 K. It is also convenient to find the speed of sound downstream of the shock $c_y = c_x\sqrt{\frac{T_y}{T_x}} = 569.6 \text{ m/s}$. The velocity $V_y$ can be found from $V_y = c_y M_y = 270.7 \text{ m/s}$ or directly from $V_y = V_x\left(\frac{\rho_x}{\rho_y}\right) = 270.7 \text{ m/s}$. Thus, we can complete the picture in the steady (shock-fixed) reference frame:




In order to transform back to the unsteady or "lab-fixed" reference frame, we need to add $V_x$ = 1044 m/s to the picture above to obtain:




Thus, the velocity of the air that has been shocked is $V_p'$ = 773.3 m/s in the unsteady reference frame. Note that the Mach number of this flow is $M_y' = \frac{V_p'}{c_y} = 1.357$, which is supersonic. This seems to contradict our conclusions in Section 7.2, where we concluded that the flow downstream of a shock wave is always subsonic. We must recall, however, that conclusion applied only to *steady* shock waves. Downstream of an *unsteady* shock, *the flow may be supersonic*. In fact, for a shock propagating into quiescent air at shock Mach numbers much greater than Mach 2, the flow will indeed be *supersonic* following the shock.

---

In this example problem, we were given the shock strength and asked to find the particle (or piston) velocity $V_p'$ behind the shock. What if we are given the converse problem: for a given particle or piston velocity $V_p'$, find the shock Mach number? This is a more challenging problem because, given $V_p'$, we cannot directly find the velocity of the shocked flow in the steady reference frame $V_y$ because we do not yet know what $V_x$ is. Further, we do not know $c_y$ because we do not know the shock strength. Thus, we cannot directly apply the normal shock relations to find the shock Mach number.

One approach to this problem is to guess a value of $M_x$ and then find $V_p'$, exactly as we did in the numerical example above. Comparing this value of $V_p'$ to the one we were given, we can iterate on $M_x$ until we find the correct shock strength.

An alternative to the tedious procedure of guessing and iterating is to start with the velocity transformation:

$$V_p' = V_s' - V_y$$

Normalizing by the speed of sound of the quiescent gas $c_x$:

$$\frac{V_p'}{c_x} = \frac{V_s'}{c_x} - \frac{V_y}{c_x}$$

It is very important to emphasize that the term $\frac{V_p'}{c_x}$ is *not* a Mach number; it is the velocity of the *shocked* gas (in the unsteady frame) normalized by the speed of sound of the *unshocked* gas. The $\frac{V_s'}{c_x}$ term *is* a Mach number, however; it is the velocity of the shock normalized by the speed of sound of the gas it is propagating into. This is simply the shock Mach number $M_s = M_x$. The $\frac{V_y}{c_x}$ term is not a Mach number, but can be expressed in terms

$\frac{V_y}{c_x} = \frac{V_y}{c_y}\frac{c_y}{c_x} = M_y\sqrt{\frac{T_y}{T_x}}$, which are a functions of $M_x$. Thus, the entire right hand side can be written as a function of $M_x$.

$$\frac{V_p'}{c_x} = M_x - M_y\sqrt{\frac{T_y}{T_x}} = M_x - \sqrt{\frac{2+(\gamma-1)M_x^2}{2\gamma M_x^2-(\gamma-1)}}\sqrt{\frac{[2\gamma M_x^2-(\gamma-1)][2+(\gamma-1)M_x^2]}{(\gamma+1)^2 M_x^2}}$$

The right hand side can be simplified to:

unsteady shock into quiescent gas

$$\boxed{\frac{V_p'}{c_x} = \frac{2}{\gamma+1}\left(\frac{M_x^2-1}{M_x}\right)}$$

This is an extremely useful relation: it relates the shock Mach number $M_x$ to the particle/piston velocity downstream of the shock normalized by the initial speed of sound $c_x$ in the quiescent gas. We must exercise caution, however, and always remember that this relation only applies for a shock propagating into *quiescent* gas. In different problems, we may need to again apply a transformation of reference frame in order to make the gas that the shock is propagating into quiescent.

This relation is a quadratic in $M_x$, so we can solve for $M_x$:

$$M_x = \frac{1}{2}\left[\frac{\gamma+1}{2}\frac{V_p'}{c_x} \pm \sqrt{\left(\frac{\gamma+1}{2}\frac{V_p'}{c_x}\right)^2+4}\right]$$

Since we are solving a quadratic, there are two solutions (note the "±" sign). Since we know the Mach number must be positive and greater than 1, we can exclude the "–" solution. Thus:

unsteady shock into quiescent gas

$$\boxed{M_x = \frac{1}{2}\left[\frac{\gamma+1}{2}\frac{V_p'}{c_x} + \sqrt{\left(\frac{\gamma+1}{2}\frac{V_p'}{c_x}\right)^2+4}\right]}$$

Note that in the limit of large $M_x$ and $V_p$':

$$\lim_{V_p'\to\infty} M_x = \frac{\gamma+1}{2}\left(\frac{V_p'}{c_x}\right)$$

Thus, in the limit of large piston and shock velocities, the shock will move a factor of $\frac{\gamma+1}{2}$ faster than the piston. For air ($\gamma = 1.4$), this means the shock driven in front of a high-velocity piston ($\frac{V_p'}{c_x} \gg 1$) will be 20% faster than the piston.

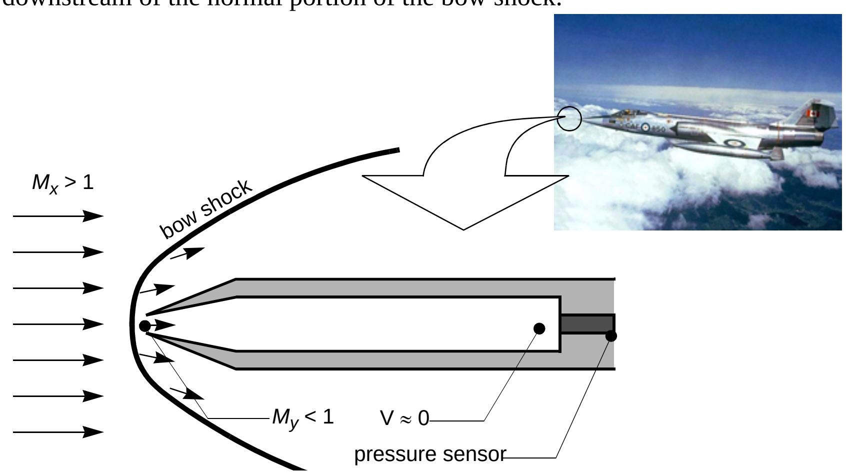
# 9. Applications of Flow with Shock Waves

We can now examine a variety of problems that involve flows with normal shock waves and isentropic flow. In all of these applications, the normal shock wave acts as a "patch" between the two regions of isentropic flow: one supersonic isentropic flow upstream of the shock, and one subsonic isentropic flow downstream.

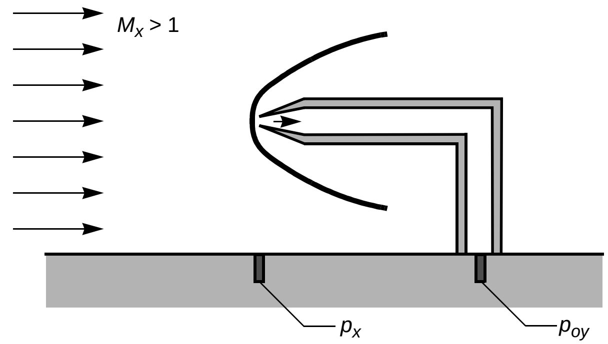
## 9.1 Pitot Probe

We have repeatedly invoked the concept of a "stagnation probe" that can sample a flow and determine its properties if brought to rest isentropically. Such probes actually exist: they are called *pitot probes*. These can be easily identified as the needle-like projections coming from the nose or wings of aircraft. A simple pitot probe is illustrated below.

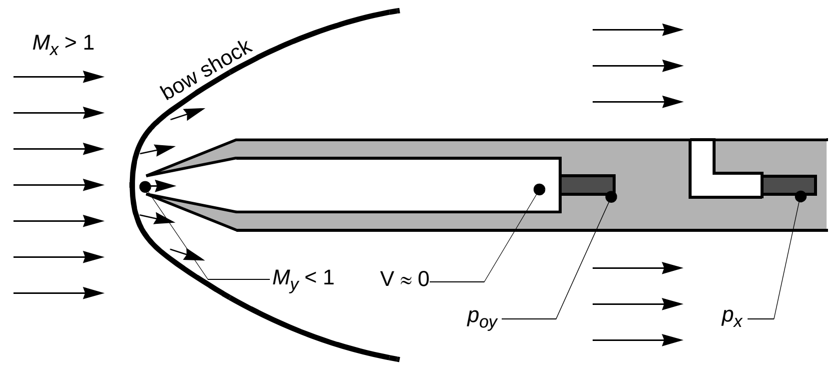
This probe can measure stagnation properties in a subsonic flow accurately. In a supersonic flow, however, a bow shock wave will appear ahead of the pitot probe. Thus, the probe will be measuring $p_{oy}$, the stagnation pressure downstream of the normal portion of the bow shock.




A pitot-static probe measures both stagnation and static pressure. In a wind tunnel, for example, a static pressure port can be located along the tunnel wall:




On an aircraft, the pitot and static probe can be combined into a single probe:




The static pressure port located along the side of the probe measures the free stream pressure upstream of the shock, $p_x$. You might think that, since the probe is behind the shock, it would measure $p_y$, the static pressure downstream of the shock. Once the flow is past the pitot tip, however, the normal shock quickly decays and the flow rapidly re-accelerates and returns almost to the freestream conditions. So, it gives an approximate measure of $p_x$, although it is nearly impossible to measure the static pressure of a supersonic flow since any measuring device placed in the flow will cause some disturbance.

For a *subsonic flow*, the pitot-static probe measures $p_o$ and $p$. These can be used to determine the Mach number via:

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

In a *supersonic flow*, the pitot-static probe measures $p_{oy}$ and $p_x$. Using the normal shock relations for $\frac{p_y}{p_x}$ and the stagnation pressure downstream of the shock:

$$\frac{p_{oy}}{p_x} = \frac{p_{oy}}{p_y}\frac{p_y}{p_x} = \left(1 + \frac{\gamma - 1}{2} M_y^2\right)^{\frac{\gamma}{\gamma - 1}} \left[\frac{2\gamma M_x^2 - (\gamma - 1)}{\gamma + 1}\right]$$

Using the relation for $M_y(M_x)$:

$$\frac{p_{oy}}{p_x} = \left[1 + \frac{\gamma - 1}{2}\left(\frac{2 + (\gamma - 1)M_x^2}{2\gamma M_x^2 - (\gamma - 1)}\right)\right]^{\frac{\gamma}{\gamma - 1}} \left[\frac{2\gamma M_x^2 - (\gamma - 1)}{\gamma + 1}\right]$$

Which can be simplified to:

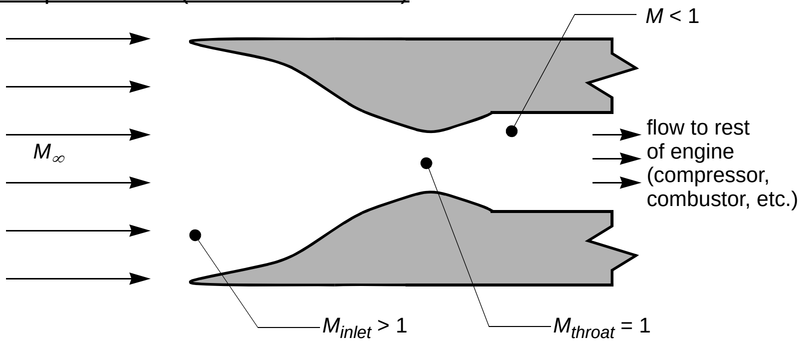
$$\frac{p_{oy}}{p_x} = \frac{\left[\frac{(\gamma + 1)M_x^2}{2}\right]^{\frac{\gamma}{\gamma - 1}}}{\left[\left(\frac{2\gamma M_x^2}{\gamma + 1}\right) - \left(\frac{\gamma - 1}{\gamma + 1}\right)\right]^{\frac{1}{\gamma - 1}}}$$

Thus, if the value of $\frac{p_{oy}}{p_x}$ can be measured, it can be directly related to the upstream Mach number $M_x$. Tables of normal shock relations (such as Table A-3) typically tabulate values of $\frac{p_{oy}}{p_x}$ or $\frac{p_x}{p_{oy}}$.

## 9.2 Supersonic Diffuser

We have discussed in some detail the exit flow of engines (nozzles, Section 6.3). We now turn to the other end of the engine: the inlet. Airbreathing engines (turbojets, ramjets, scramjets, etc.) require that the air taken into the engine be decelerated and compressed (i.e., “diffused”) before being fed into the combustor. Deceleration is required because it is extremely difficult to mix and burn fuel in a stream of high velocity air, and because the thermodynamic cycles of engines (e.g., Brayton cycle) requires that the flow be compressed prior to heat addition in order for efficient operation. For *subsonic* engines, this “diffusion” (deceleration and compression) of the flow can be accomplished by an increase in area and the action of compressor blades (these are the “propellers” that you see if you look into the intake of a turbojet engine). For an engine intended to operate in *supersonic* flight, designing a diffuser to decelerate the supersonic flow is an interesting challenge.

The first concept for a supersonic diffuser is simply to run the converging diverging nozzle backwards. In this case, the supersonic flow entering the inlet encounters a converging area, which decelerates the flow and drives it towards Mach 1. If the diffuser has the correct area ratio, the flow will reach Mach 1 at the throat, and then can continue to decelerate *subsonically*, feeding a low speed and high pressure flow into the combustor section of the engine (what we want). This is nothing more than the converging-diverging nozzle we have seen before, but run backwards. You might wonder why the flow does not re-accelerate on the *supersonic* branch of the solution when it enters the diverging section after the throat. This is possible, but the rest of the engine (combustion, nozzle, etc.) acts as the “receiving chamber” that determines the back pressure. Since the combustor typically operates at high pressure compared to the free isentropic diffuser (backwards nozzle)



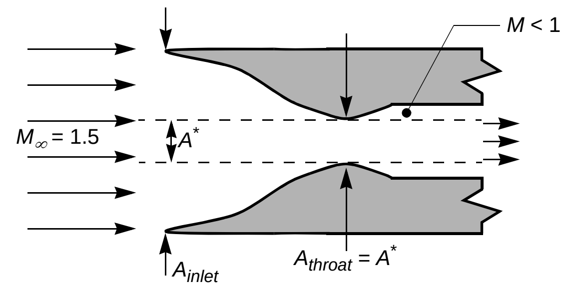

stream static pressure, it will force the flow to expand along the subsonic branch.

This design of a diffuser is very efficient. In principle, it can be operated isentropically, so that there is no loss in stagnation pressure. As we saw in Section 7.2, the loss in stagnation pressure is directly related to an increase in entropy, and a decrease in the overall efficiency of the device. Therefore, this isentropic diffuser is the most efficient diffuser design possible. Unfortunately, it is also very unstable. Consider a diffuser optimized for a flight Mach number of $M_\infty = 1.5$. Since the throat must be sonic, we know $A_{\text{throat}} = A^*$. The flow at the inlet is at Mach 1.5, and since the entire flow is isentropic, we know that

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\left(\frac{A}{A^*}\right)_{M\,=\,1.5} = 1.1762$$

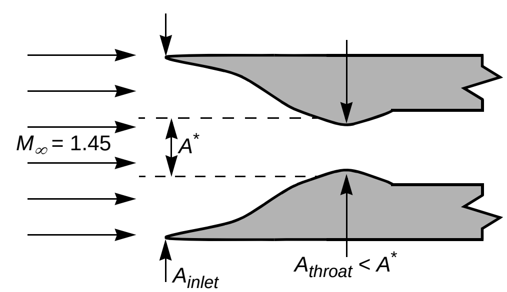
Therefore, $\dfrac{A_{\text{throat}}}{A_{\text{inlet}}} = 0.850$. If we recall that sonic area $A^*$ is a reference area that is carried along with the flow and remains constant, we can represent it as a dashed streamtube in the flow.




Now imagine what happens if the flight Mach number were to decrease slightly to, say, $M = 1.45$, due to a slight disturbance (maybe the pilot bumps the throttle back, oscillation in engine thrust, etc.). The area ratio of the diffuser $\dfrac{A_{\text{throat}}}{A_{\text{inlet}}} = 0.85$ remains fixed, but the sonic area changes:

$$\left(\frac{A}{A^*}\right)_{M\,=\,1.45} = 1.1440$$

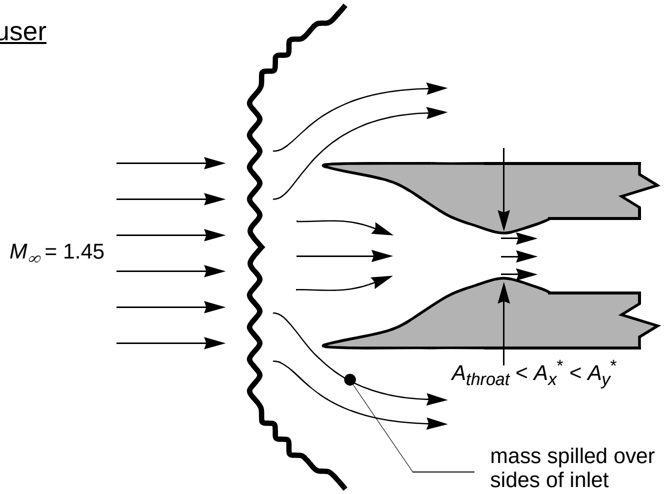
Thus, $A^* = 0.874\, A_{\text{inlet}}$, which is *larger* than the physical throat of the diffuser: $A_{throat} = 0.850\, A_{inlet}$. The picture in this case becomes:




Recall that, in an isentropic flow, the flow can *never* pass through an area smaller than the sonic area. Yet, if we operate this diffuser optimized for

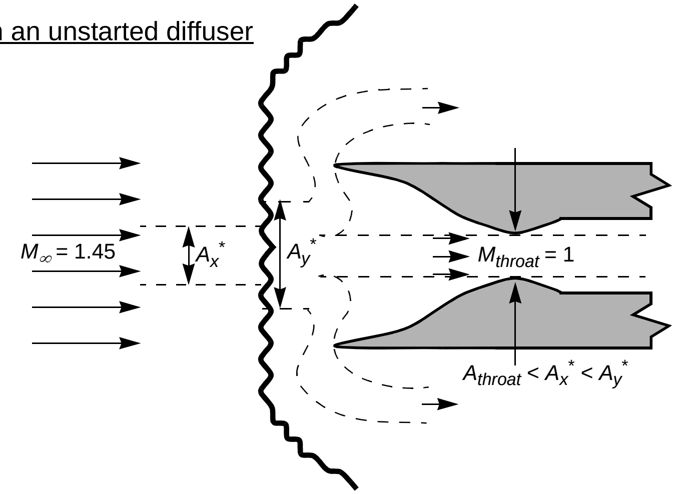
Mach 1.5 at Mach 1.45, that is exactly what we are trying to do: we are trying to force a flow though a throat smaller than the sonic area. It is as if we are attempting to pass too much mass flow through too small of an opening. Since the flow is supersonic, there is no way to communicate upstream through the isentropic flow that the throat is too small. The way Nature resolves this situation is similar to what happens if a person tries to ingest too much food: a normal shock is "vomited" or "disgorged" out of the diffuser and it will position itself upstream of the inlet.

unstarted diffuser




Note that, since sonic area increases across a normal shock ($A_y^* > A_x^*$), the sonic area downstream of the shock is now even larger than before. Since the flow downstream of the shock is subsonic, however, it can now adjust itself to the fact that the throat is too small, and some of the flow will be spilled over the side. The flow at the diffuser throat will be choked, meaning that it is accepting as much mass flow as possible.

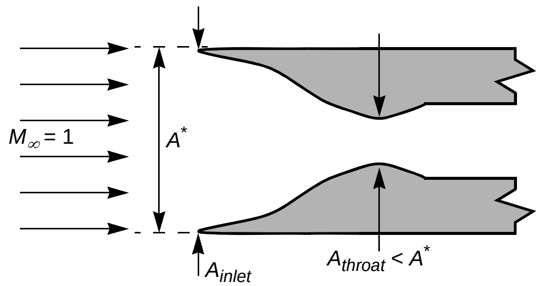
sonic area in an unstarted diffuser




This process of disgorging the shock wave is called an "unstart" and is a very violent process that should be avoided at all costs. Once the inlet is unstarted, the strong normal shock in front of the inlet will result in stagnation pressure losses: $\left(\frac{p_{oy}}{p_{ox}}\right)_{M_x = 1.45} = 0.9448$, resulting in inefficient engine operation. Also, the spillage of flow over the sides of the engine will lower thrust (thrust is typically proportional to $\dot{m}$ for most engines).

What is worse, if we have a diffuser optimized for a supersonic flight Mach number, it will already be unstarted once the flow becomes supersonic. Consider a diffuser on a supersonic jet plane: initially, of course, the flow is subsonic into the inlet as the jet takes off and accelerates toward the sound barrier. Once it reaches supersonic speed, consider the flow into the inlet:

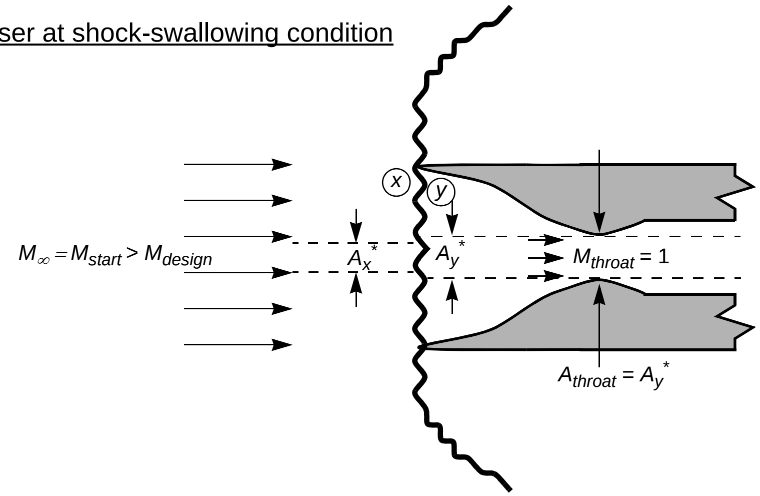
diffuser at sonic flight Mach number




Since the flow is already nearly sonic at the inlet, $A_{\text{inlet}} = A^*$. Thus, the physical throat is already too small, and a normal shock will form ahead of the inlet as soon as the flight Mach number becomes supersonic. If the plane is able to continue accelerating, the normal shock will remain in front of the inlet, increasing in strength and resulting in increasing stagnation pressure losses. Even if the plane can reach the design Mach number, it is not started because, as we saw above, the sonic area increases across the shock. Thus, the diffuser is "unstarted" before it was even "started" to begin with.

In order to "start" the flow though the diffuser (i.e., to establish supersonic flow through the diffuser), it is necessary to *overspeed* the diffuser. As we increase the flight Mach number, $A_y^*$ continues to decrease (even though the ratio $\dfrac{A_y^*}{A_x^*}$ is increasing) until eventually the diffuser can accept the entire post-shock flow:

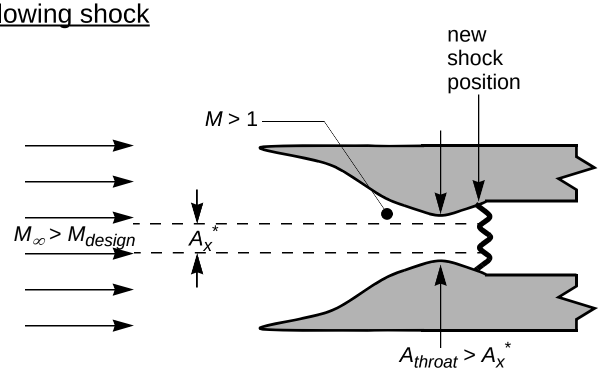
diffuser at shock-swallowing condition




The normal shock in this case will be located right at the inlet lip, since no flow is spilled over the side. If the flight Mach number is further increased slightly, the normal shock will be pushed into the diffuser, where it is unstable, and it will be immediately "swallowed" or "ingested" through the diffuser. We would now say that our inlet is *started*. The shock swallowing process is extremely fast (typically, milliseconds) and is an unsteady flow that we cannot properly treat using our quasi-steady analysis. The shock will ultimately re-position itself at a location dictated by the back pressure conditions provided by the combustor and the rest of the engine.

For our example of the inlet optimized for Mach 1.5, where $A_{throat} = 0.850\, A_{inlet}$, we can solve for what this starting Mach number must be. Since the throat is now equal to the sonic area downstream of the shock, $A_{throat} = A_y^*$. Thus, $\dfrac{A_y}{A_y^*} = 1.176 \Rightarrow M_y = 0.61$. Using the normal shock relations, $M_x = 1.83$. Thus, it is necessary to overspeed the diffuser to Mach 1.83 in order to swallow the shock and start the diffuser.

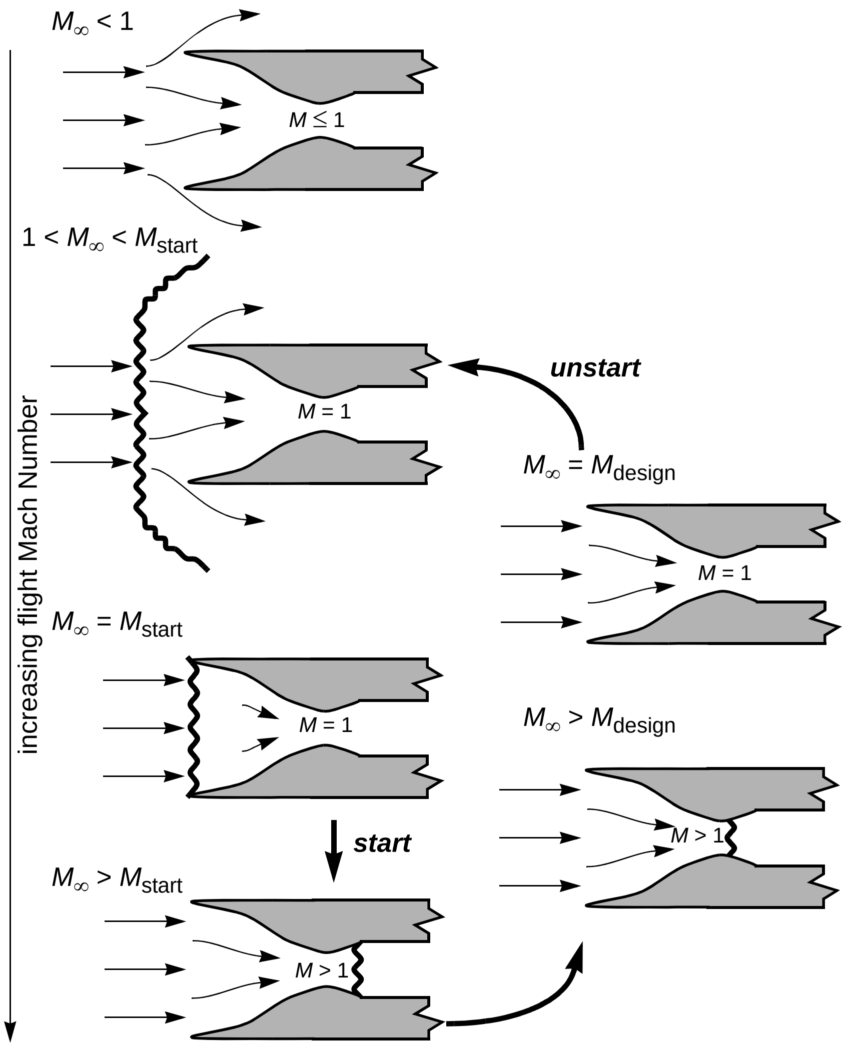
diffuser after swallowing shock



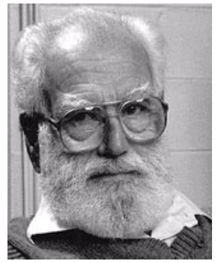

Once started, the jet can slow back down to Mach 1.5 in order to operate the ideal, isentropic diffuser. If the vehicle should decelerate even slightly below this design case, however, the diffuser will unstart, and it will be necessary to accelerate back up to Mach 1.83 in order to start the flow. For this reason, most supersonic diffusers operate above the ideal isentropic limit ($M > M_{\mathrm{design}}$), so that the engine can recover from a perturbation in flight speed, etc., without resulting in an unstart. Operating at these conditions means that the throat area is slightly larger than the sonic area and therefore a shock wave will be located downstream of the throat. By proper tuning of the back pressure, however, the normal shock can be kept near the throat where the flow Mach number is low and therefore stagnation pressure losses are small.

Thus, we see that operation of a supersonic diffuser exhibits an interesting *hysteresis* phenomenon: between Mach numbers of $M_{\mathrm{design}} < M < M_{\mathrm{start}}$, the diffuser can operate in *two* possible configurations, depending on whether it is coming from the "unstarted" or "started" configuration.

The entire sequence of starting a supersonic diffuser with a fixed area ratio by varying the flight Mach number is illustrated here.



This requirement for starting a supersonic diffuser was first identified during the Second World War by Arthur Kantrowitz and Coleman duPont Donaldson in a report that was classified at the time.* The Mach number required to start a diffuser of a given area ratio is called the "Kantrowitz Limit."



Arthur Kantrowitz

We can derive relations to describe both the isentropic design case and the Kantrowitz starting limit:

$$\left(\frac{A_{throat}}{A_{inlet}}\right)_{design} = M_{design}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_{design}^2\right)\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

This is simply the reciprocal of the familiar $\frac{A}{A^*}$ relation.

For the Kantrowitz starting condition:

$$\left(\frac{A_{throat}}{A_{inlet}}\right)_{start} = \frac{A_y^*}{A_{inlet}} = M_{ystart}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_{ystart}^2\right)\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$M_{ystart}$ is related to $M_{xstart}$ by the normal shock relation:

$$M_y = \sqrt{\frac{2+(\gamma-1)M_x^2}{2\gamma M_x^2-(\gamma-1)}}$$

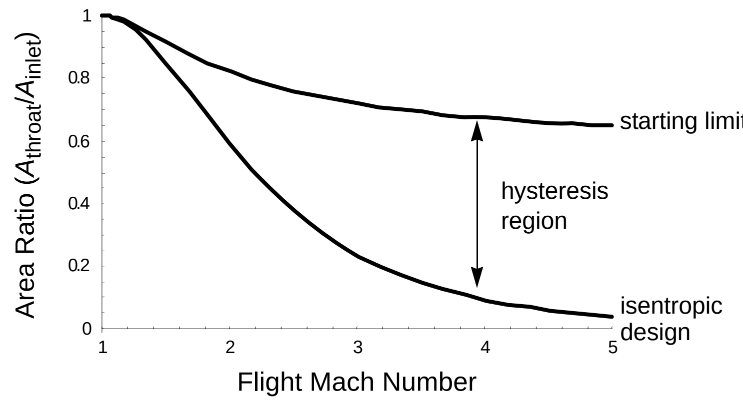
Substituting:

$$\left(\frac{A_{throat}}{A_{inlet}}\right)_{start} = \sqrt{\frac{2+(\gamma-1)M_{start}^2}{2\gamma M_{start}^2-(\gamma-1)}}\left\{\left(\frac{2}{\gamma+1}\right)\left[1+\frac{\gamma-1}{2}\left(\frac{2+(\gamma-1)M_{start}^2}{2\gamma M_{start}^2-(\gamma-1)}\right)\right]\right\}^{-\frac{\gamma+1}{2(\gamma-1)}}$$

Which can be simplified to:

$$\left(\frac{A_{throat}}{A_{inlet}}\right)_{start} = \sqrt{\frac{2+(\gamma-1)M_{start}^2}{1+\gamma(2M_{start}^2-1)}}\left[\frac{(\gamma+1)M_{start}^2}{1+\gamma(2M_{start}^2-1)}\right]^{-\frac{1}{2}\frac{\gamma+1}{\gamma-1}}$$

We can plot both of these relations versus the freestream Mach number.




The region of hysteresis is between these two curves. For a given area ratio, an inlet operating in the Mach number regime between these two curves can either be started or unstarted. Note that, in the case of diffusers designed for flight Mach numbers greater than Mach 2, it becomes impossible to start a fixed area ratio diffuser by overspeeding. This is because, even in the limit as $M_x \to \infty$, $M_y \to 0.378$ as we saw in Section 7.3. At this limit, $\frac{A}{A_y^*} = 1.666$.

This inlet area ratio is optimized for a flight Mach number of $M_x = 1.98$. Thus, for inlets optimized for higher Mach numbers, it is not possible to start the diffuser by overspeeding.* For supersonic diffuser designed for $M > 2$, it is necessary to use a *variable-area inlet* in order to start the diffuser.

*Kantrowitz, A., and Donaldson, C., "Preliminary Investigation of Supersonic Diffusers," NACA WRL-713, 1945.

To illustrate the variable-area diffuser approach, let us return to the case of our diffuser designed to operate at Mach 1.5. Recall that, if the diffuser is accelerated from subsonic flow to supersonic flow, a normal shock will already exist in front of the inlet when it reaches the design flight Mach number $M_{design} = 1.5$. If the diffuser area ratio is fixed, it is necessary to accelerate to $M_{start} = 1.83$ in order to swallow the normal shock. It may be exceedingly difficult or impossible to continue to accelerate to this speed with an unstarted inlet. However, we can start at the design flight Mach number if the throat is opened up to accept the full sonic area $A_y^*$. In this case, the shock will be pulled back to the lip of the inlet. Any further increase in throat area (or flight Mach number) will result in the shock being swallowed, and supersonic flow will be established through the diffuser.

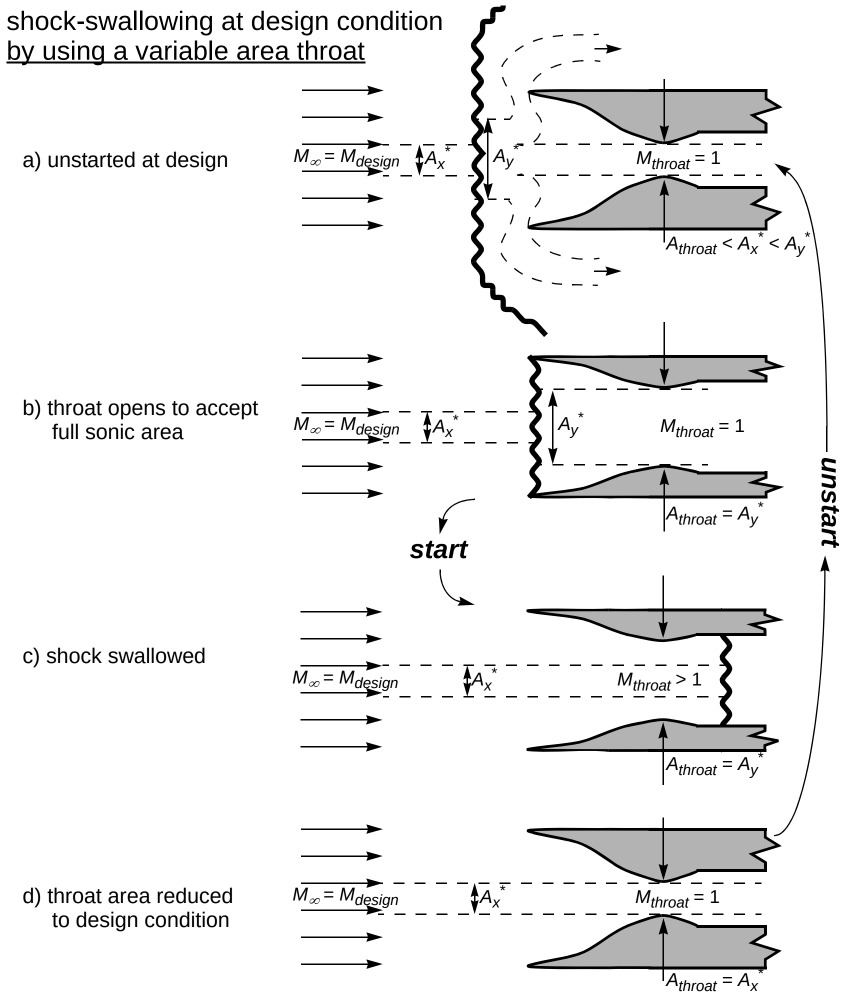
At our design case of $M_{design} = 1.5$, we can find the throat to inlet area ratio required to swallow the shock:

$$\frac{A_{throat}}{A_{inlet}} = \frac{A_y^*}{A_y}$$

At the design Mach number of $M_{design} = 1.5$, the Mach number downstream of a normal shock is $M_y = 0.7011$.

$$\left(\frac{A}{A^*}\right)_{M = 0.7} = 1.0944$$

$$\frac{A_{throat}}{A_{inlet}} = \left(\frac{A}{A^*}\right)^{-1} = 0.914$$




Thus, if the throat can open up from 85% of the inlet area to 91% of the inlet area, then the shock can be swallowed at the design Mach number, without having to accelerate further. Once swallowed, the throat area can be reduced back to $A_{throat} = 0.85\ A_{inlet}$ without unstarting. If the throat area is reduced any further (or if flight Mach number decreases), the diffuser will unstart and the throat would need to be re-opened in order to re-establish supersonic flow.

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[*] Interestingly, it *is* possible to start an inlet below the Kantrowitz limit in an *unsteady*, impulsively started flow. This phenomenon does not have much relevance to flight vehicles that must accelerate continuously from subsonic Mach numbers, but it does permit diffusers in unsteady shock-tunnels and impulsively-launched projectiles to be started under conditions that would be impossible in steady flow.

Again, most real diffusers would be operated slightly away from the "hairy edge" of the optimal design condition.

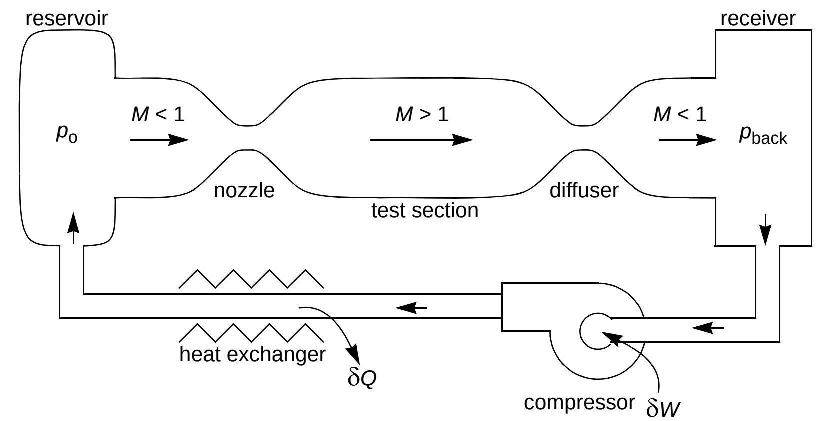
Note there is also a hysteresis present for variable area inlets as well: at a given flight Mach number and area ratio, the flow may be started or unstarted if we are in the region between the "starting" and "design" curves. Whether the flow is started or unstarted at a given Mach number will depend on if the throat area has been decreased from the starting condition at that Mach number or, on the other hand, if the diffuser is already unstarted.

Note that, even with a variable area inlet, the diffuser will not likely be started as it first encounters supersonic flight ($M > 1$). Thus, all diffusers are "unstarted" until they reach a Mach number at which they can be started. Recall from Section 7.4, however, that the stagnation pressure loss across weak shock waves is very small. In fact, below approximately Mach 1.3, a normal shock is almost isentropic. Thus, it is possible for a diffuser to operate efficiently without starting at low supersonic Mach numbers. In fact, some supersonic jet engines are designed to use a "normal shock diffuser," which is acceptable for $M < 1.3$. As Mach number increases, however, losses in stagnation pressure increase dramatically for a normal shock, and it is essential to establish supersonic, nearly isentropic flow through the diffuser for efficient operation.

## 9.3 Supersonic Windtunnel

Supersonic flows can be created in the laboratory by use of a wind tunnel. A continuous wind tunnel that recirculates the flow through the test section is shown here. This design has the advantage of long test times compared to transient blow-down or pulsed wind tunnel designs. In this section, we will examine the operation of a continuous flow wind tunnel.

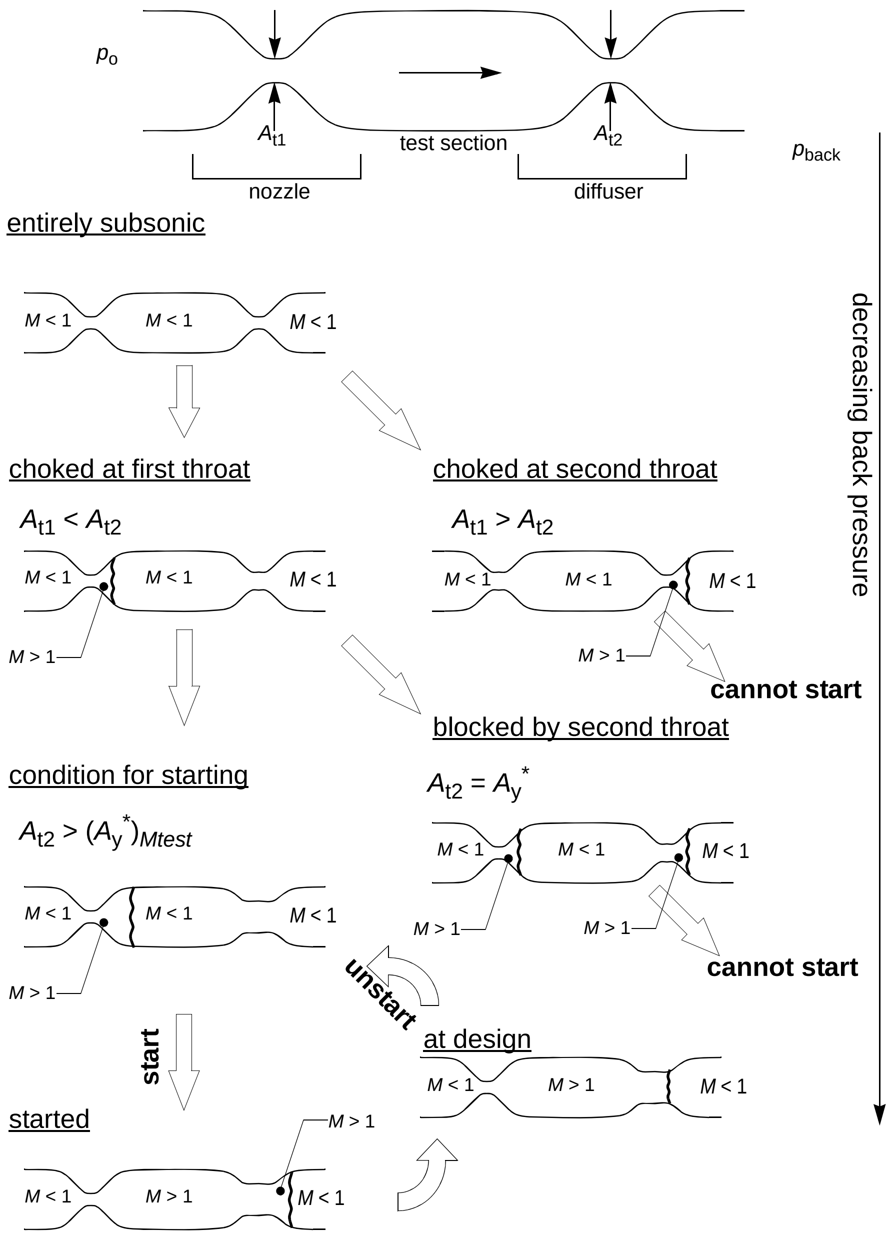
This windtunnel generates supersonic flow from a reservoir of high pressure gas via a converging-diverging nozzle. The supersonic flow passes through the test section (where the model of the test article would be mounted), and is then decelerated back to subsonic speeds by a diffuser, similar to the diffuser of supersonic inlets discussed in the prior section. This diffuser is critical to the operation of the windtunnel: unless the flow is decelerated to subsonic speeds,




it is necessary to maintain the back pressure at a very low value compared to $p_o$ in order for supersonic flow to exist in the test section (recall the discussion of converging-diverging nozzles in Section 6.2). Further, it is necessary to bring the flow back to near rest in order for it to be compressed back to the reservoir conditions $p_o$. The more nearly isentropic the diffuser can operate, the less work input is required into the compressor. Note that, in principle, if the wind tunnel can be operated entirely isentropically, it is possible for the back pressure in the receiving chamber $p_{back}$ to nearly equal to the reservoir pressure $p_o$, and the wind tunnel would run like a perpetual motion machine. Of course, perfectly isentropic operation is not possible, and therefore it is necessary to input work via the compressor in order to offset stagnation pressure losses due to friction and shock waves. Typically, a heat exchanger is also required to remove heat from the flow before being sent back through the test section.

In order to operate as intended, supersonic flow must first be established through the test section; in other words, the tunnel must be "started", similar to the starting of the supersonic inlet in Section 9.2.

The starting process begins with the flow quiescent and at stagnation pressure $p_o$. As the back pressure decreases, flow is induced through the wind tunnel. If the back pressure is reduced sufficiently, the flow will choke at the smaller of the two throats ($A_{t1}$ or $A_{t2}$). If the throat at the diffuser chokes first ($A_{t2} < A_{t1}$), then reducing the back pressure further cannot affect the flow




though the test section; the choking of the second throat has isolated the tunnel from the back pressure and the tunnel cannot be started. If the throat of the nozzle chokes ($A_{t1} < A_{t2}$), then lowering the back pressure will result in the flow becoming supersonic in the nozzle, followed by a normal shock. Once the flow is choked *at any point*, the mass flow rate through the windtunnel is fixed and will not change as $p_{back}$ is reduced further.

For the case of the first throat choking, reducing the back pressure further will result in the normal shock receding down the nozzle toward the test section. As the flow velocity in the test section increase, it may be possible that the second throat will still choke, even though it is physically larger than the first throat ($A_{t1} < A_{t2}$). This is because sonic area increases across the shock. As the shock approaches the test section, the shock becomes stronger and $A_y^*$ increases. If the second throat is not large enough to pass this area when the shock is in the test section, it will choke and the windtunnel is *blocked*. In this case, lowering the back pressure further will not result in the establishment of supersonic flow in the test section.

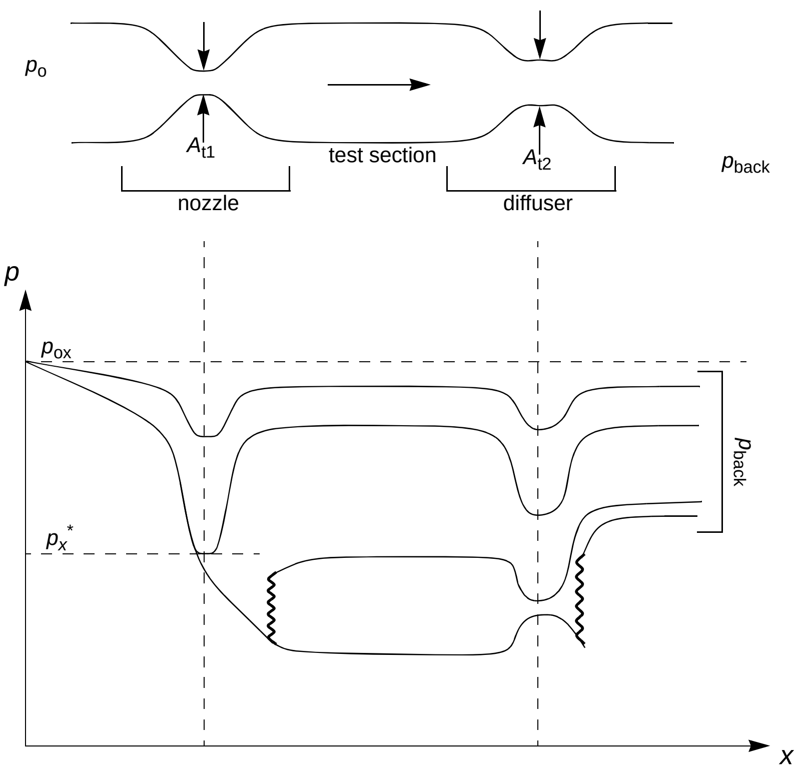
The requirement for starting is that the second throat must be large enough to accept the sonic area $A_y^*$ when the shock has receded back to the entrance of the constant area test section:

$$A_{t2} \ge \left(\frac{A_y^*}{A_x^*}\right)_{Mtest} A_{t1}$$

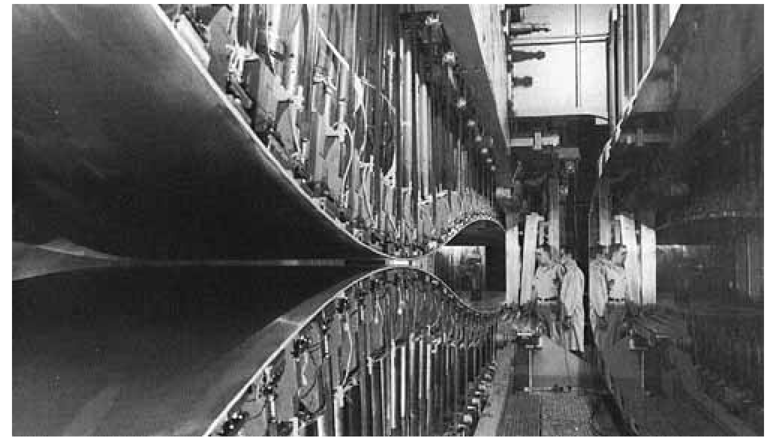
Note in this case, the Mach number of the shock is the Mach number that the test section is designed to produce. Since the flow downstream of the shock is entirely subsonic, the shock can continue to be influenced by the back pressure. If the back pressure is reduced any further, the normal shock will enter the test section, where it is unstable. The shock will rapidly snap through the test section and appear downstream of the second throat in the diffuser. At this point, the tunnel is *started*.

Similar to the hysteresis in a supersonic inlet, once the tunnel is started, the back pressure can be increased in order to move the shock closer to the diffuser throat in order to make the shock weaker and reduce total pressure losses. If the shock is forced past the throat of the diffuser, however, the shock will be forced upstream, through the test section and into the nozzle. The tunnel will be *unstarted*, and the back pressure will need to be reduced again in order to re-establish supersonic flow.

The pressure profiles for the starting process of a supersonic windtunnel that fulfills the starting criterion is illustrated here.




The starting process is greatly facilitated if a variable area throat can be used. In this case, the diffuser throat can be opened up for the purposes of starting the tunnel, and then reduced in area to bring the diffuser closer to near-shock free (i.e., isentropic) deceleration of the flow.



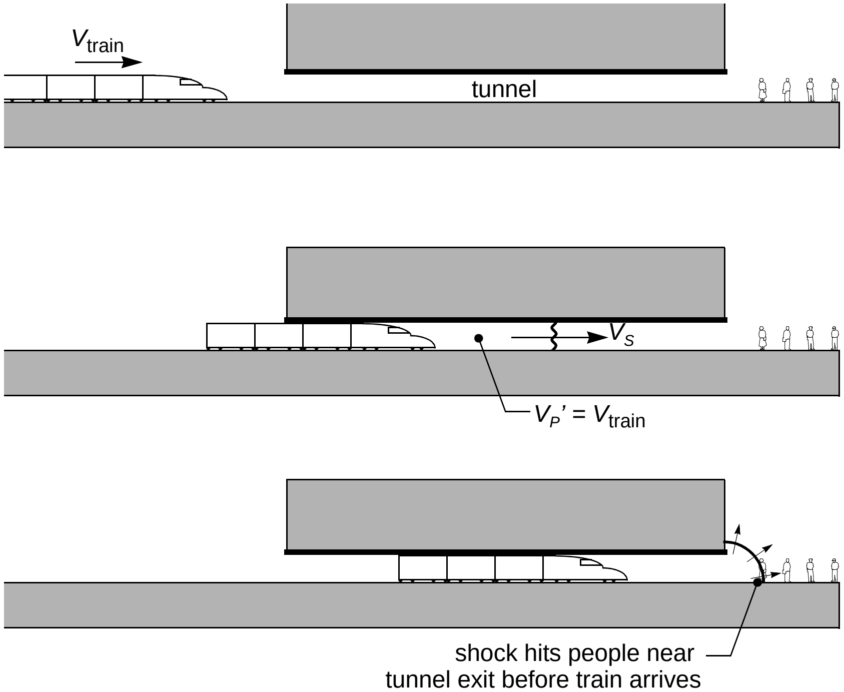
A supersonic wind tunnel nozzle at Arnold Engineering Development Center (von Karman Facility) showing the mechanism that permits the area profile of the tunnel to be varied.

## 9.4 Shinkansen Train in a Tunnel

When a high-speed train enters a tunnel, a strong compression wave is generated that travels down the tunnel in front of the train. This compression wave can steepen into a shock wave that, upon emerging from the far end of the tunnel, can create a “sonic boom” which is particularly unpleasant to people in the vicinity. In Japan, high-speed Shinkansen or “bullet trains” are continuously increasing their operational velocity (now approaching 300 km/hr), and this only exacerbates the shock wave phenomenon in tunnels. In this section, we will perform a highly simplified analysis of this problem to provide some bounds to how strong these train-generated shocks can be and suggest possible means to mitigate the problem.

To begin with, consider a Shinkansen train traveling at 300 km/hr (or 83.3 m/s). If the train is traveling through open country, the flow approaching the train is at a Mach number of $M = \frac{V_{train}}{c_{ambient}} = \frac{83.8\text{ m/s}}{348\text{ m/s}} = 0.24$. This is clearly in the

Shinkansen "Bullet Trains"




subsonic, incompressible regime, where we can safely neglect compressible effects and shock waves should not occur.

Now, consider what happens when such a train enters a tunnel that exactly matches the cross sectional area of the train: the train becomes like a piston in a cylinder. The air in the tunnel has no way to slip around the train, so it must pile-up in a slug of gas moving at the train's velocity. This means the train is imposing an instantaneous particle velocity of $V_p' = V_{train}$ = 83.3 m/s upon the air at the entrance of the tunnel. This will result in a shock wave propagating down the tunnel in front of the train. The shock wave, in effect, is a wave that is communicating the approach of the train, and forms the boundary between air that is quiescent and air that has been impulsively accelerated to $V_p' = V_{train}$. Because the slug of air will get longer and longer as the train pushes more air in front of it, the shock will have a velocity greater than the train. In fact, the shock (indeed, all shock waves) must be supersonic with respect to the air it propagates into.

How strong will this shock be? Knowing the particle velocity and the sound speed of the quiescent gas the shock is propagating into, we can solve for the shock strength from (see Section 8.4):

$$M_s = \frac{1}{2}\left[\frac{\gamma+1}{2}\frac{V_p'}{c_x} + \sqrt{\left(\frac{\gamma+1}{2}\frac{V_p'}{c_x}\right)^2 + 4}\right]$$

$$M_s = 1.154$$

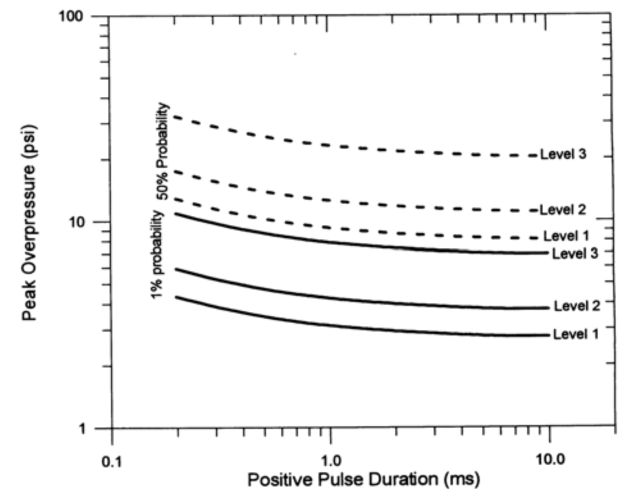
In other words, even though the train is only going Mach 0.24 when in open country, if it enters a tight fitting tunnel, it will drive a Mach 1.154 shock wave down the tunnel in front of it. The strength of this shock comes directly from the normal shock relations:

$$M_x = 1.154 \Rightarrow \frac{p_y}{p_x} = 1.39$$

Or a pressure increase of 39%. This overpressure can be strong enough to do damage and cause injury to structures and personnel located near the tunnel exit. For example, a pressure increase of 39% is equal to an overpressure of 5.7 psi. As seen in the figure, if this pressure is applied to a subject for more than 1 millisecond (which will likely be the case for a long tunnel), it can result in permanent hearing loss in 1% of people exposed and may even result in ear drum rupture and inner ear damage. This is clearly unacceptable!

If we want to limit the potential damage to minor, temporary hearing loss in 1% of cases, then the over pressure must be kept below 3 psi. This would correspond to a shock strength of $\frac{p_y}{p_x} = \frac{14.7 + 3\text{ psi}}{14.7\text{ psi}} = 1.204$. This corresponds to a shock Mach number of $M_s = 1.082$. This, in turn, corresponds to a particle velocity of:

$$\frac{V_p'}{c_x} = \frac{2}{\gamma+1}\frac{M_x^2 - 1}{M_x} = 0.1315$$



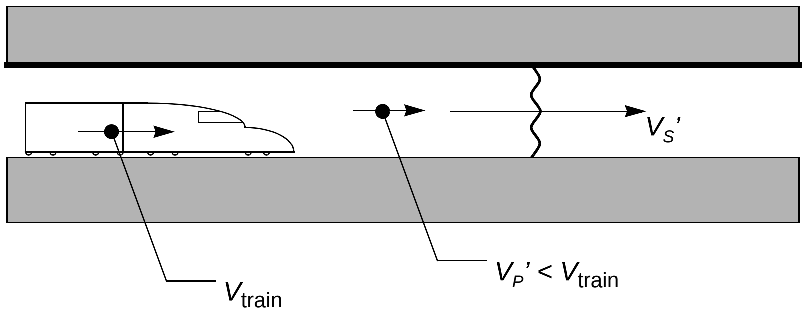

Damage caused to ears by a single pressure pulse (shock).*
Level 1: Minor damage and/or healable small tears or rupture of the tympanic membrane (ear drum)
Level 2: Tearing of the membrane that will result in permanent hearing loss.
Level 3: Severe rupture of the membrane along with inner ear damage.

*Taken from: Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, 1996, pp. 419-420.

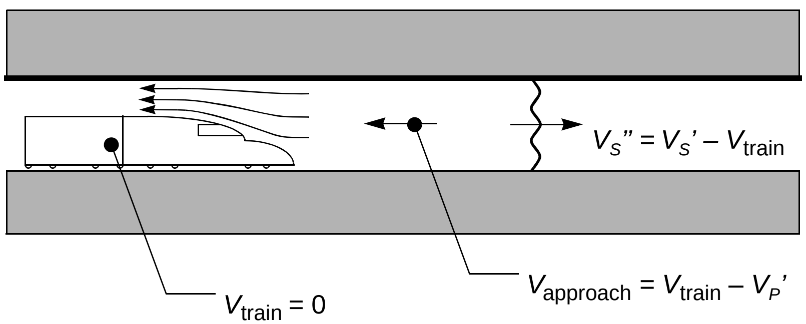
Thus, $V_p' = 0.1315\, c_x = 45.8$ m/s, or 165 km/hr. If the train stays below this speed, then the shock will be of an acceptable strength. But, this is well below the operational speed of many Shinkansen trains, so this is also an undesirable solution.

An obvious solution is to increase the diameter of the tunnel, opening up a gap between the train and the tunnel wall to permit the air to squeeze past. Note that now the train does not match the particle velocity $V_p'$ behind the shock; the train has now become a “leaky piston” so that some of the shocked air can squeeze past the train. From the point of view of a passenger on the train, they

In "ground-fixed" frame



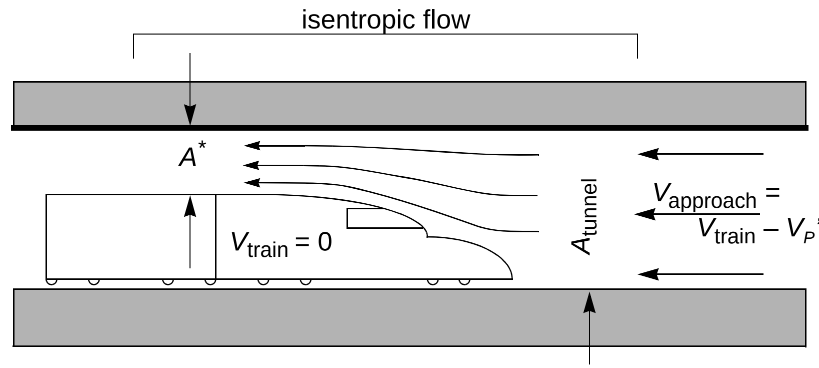

In "train-fixed" frame




would see air approaching the train at a velocity $V_{approach} = V_{train} - V_P'$. How much air will get past the train? The flow over the top of the train will be *choked*, so that as much air as possible is getting past. We can see this is the case because, if the flow was not choked, it would be entirely subsonic, and the fact that more air can pass over the train would be communicated to the shock and it would decelerate until the flow does choke. Although the shock is not steady in this train-fixed reference frame (it is moving at $V_S'' = V_S' - V_{train}$), the flow approaching the train is steady, so we may model it using steady isentropic flow relations:

If we fix the shock strength at $M_S = 1.082$ (i.e., shock causing 1% of "Level 1" damage) and the train velocity at 300 km/hr (83.3 m/s), the velocity at which flow approaches the train is:

In "train-fixed" frame




$$V_{approach} = V_{train} - V_P' = 83.3 - 45.8 \text{ m/s} = 37.5 \text{ m/s}.$$

To express this approach velocity as a Mach number, we need the speed of sound downstream of the shock:

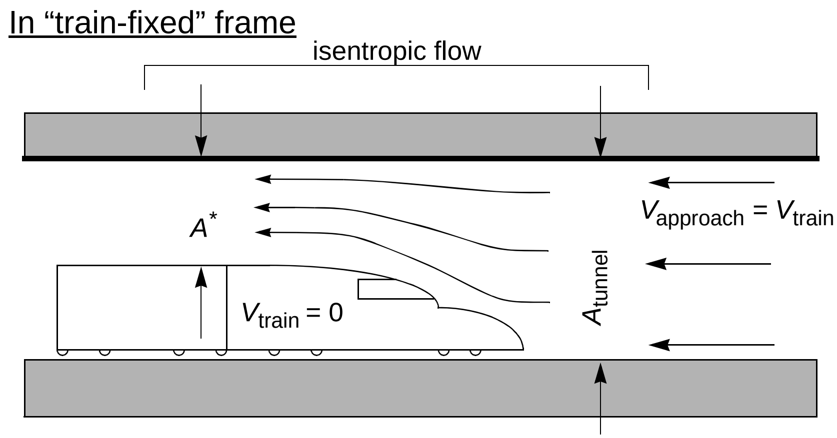
$$c_y = c_x\sqrt{\frac{T_y}{T_x}} = 348 \text{ m/s}\sqrt{1.053} = 357 \text{ m/s}$$

So, the Mach number of the flow approaching the train is $M_{approach} = \frac{V_{approach}}{c_y} = 0.105$. For this Mach number, the isentropic flow relations give an area ratio: $\frac{A}{A^*} = 5.56$. Since $A^*$ is the area between the train and the tunnel:

$$\frac{A_{tunnel}}{A^*} = \frac{A_{tunnel}}{A_{tunnel} - A_{train}} = 5.56 \Rightarrow A_{tunnel} = 1.22A_{train}$$

So, if the tunnel is 22% larger in cross-sectional area than the train, enough air will squeeze by the train to maintain the shock strength at a reasonable level.

If we increase the tunnel area further, we can eliminate the shock wave altogether. If the area between train and tunnel wall is large enough to accept the entire flow approaching the train as it enters the tunnel, isentropic flow would predict that no shock will form.




Since the train is traveling at a Mach number of $M_{train} = 0.24$ with respect to the quiescent air, the sonic area for this Mach number is:

$$\left(\frac{A_{tunnel}}{A^*}\right)_{M\,=\,0.24} = 2.50 \Rightarrow A_{tunnel} = 1.66A_{train}$$

Thus, isentropic flow predicts that, if the tunnel is 66% larger in area than the train, no shock wave should form at all. Since area scales as the square of diameter, the tunnel needs to be only $\sqrt{1.66} = 1.29$, or 29% large in diameter than the train. Most tunnels already meet this criterion.

Despite these calculations, the "sonic boom" problem of high-speed trains in tunnels persists today, even though the tunnel-to-train area ratios used are much larger than those discussed here. The shock wave strengths are also much smaller, but any population subjected to unexpected shock waves of *any* finite strength finds them an extremely startling and unpleasant phenomenon; the sonic boom problem has all but extinguished discussion of supersonic civilian transports such as the Concorde. The compression wave phenomenon persists in tunnels even when our isentropic flow relations predict that there should be no shock wave because the flow induced by the motion of the train through the tunnel is considerably more complex than our simple model can handle. Nonetheless, the basic considerations and trends noted here (i.e., increasing tunnel area mitigates shock strength) still apply. Current research focuses on modifying the geometry of the tunnel entrance and exit by introducing dampers (similar to the silencer on a gun) to alleviate this problem.

## 9.5 Caveats

Similar to rocket nozzles and shocks in converging-diverging nozzles, a number of complicating factor are present in the applications we examined in this chapter. Namely, the presence of boundary layers resulting from viscosity and multi-dimensional effects resulting from the fact that the variation in area is not always gradual in these applications means that our analysis is not perfect. In supersonic diffusers, in particular, the fact that pressure increases continuously through the diffuser means that there is an adverse pressure gradient acting against the boundary layer. Thus, flow separation is an important concern in supersonic inlet design that must be taken into account in designing a diffuser. The presence of the adverse pressure gradient means that, all other things being equal, diffuser design is much more difficult than nozzle design. Realizing isentropic deceleration of a supersonic flow is also a practical impossibility. In reality, a supersonic flow encountering a converging area will result in a system of oblique shock waves and expansions that will be treated in Chapters 12 and 13. The resulting diffuser flow is multidimensional, and considerable care must be exercised in designing the diffuser geometry.

All of these same considerations apply in supersonic windtunnels as well, if not more so. The continuous build up of boundary layers on the wall of a supersonic wind tunnel is an issue of enormous concern to aerodynamicists who design and use the facilities. The fact that viscous effects become more significant at a smaller scale (recall Reynolds number: $\mathrm{Re} = \frac{\rho V L}{\mu}$) has consistently driven up the size (and cost) of supersonic wind tunnels, where "bigger is better" is the general rule.

Despite all these complicating factors, the general trends we identified here are still valid and the quantitative results are still remarkably accurate. The fact that engineers continue to use these same tools of quasi-steady one-dimensional flow in the design of diffusers and windtunnels is a testament to the power of using an analysis that derives from the basic conservation laws.

# 10. Frictional Flow

We now turn our attention to flow in which friction between the channel wall and the flowing gas is present. The development of our analysis of frictional flow (also called "Fanno Flow") is similar to how we developed isentropic flow (Chapter 5). We will first examine differential relations that tell us how flow responds to friction over an infinitesimal length of flow. Then, we will develop integral relations that apply across finite length channels. Finally, we will look at how flow responds to changes in upstream and downstream pressure (stagnation and back pressure). The relations developed will be analogous to those for isentropic flow. A major difference, however, is that the action of friction results in an irreversibility, so entropy will increase. This, in turn, results in a decrease in stagnation pressure ($p_o$), so we must exercise caution in using $p_o$ as a reference state ($p_o$ will no longer be constant throughout a flow). In this chapter, we will consider that the flow is still adiabatic, however, so $h_o$ and $T_o$ are still constant. In the next chapter, we will examine flow with heat transfer (called "Rayleigh Flow"), in which both $p_o$ and $T_o$ will vary due to heat addition (or removal). This table compares these types of flow to isentropic flow:

| Type of Flow | Factor Causing Change | $p_o$ | $T_o$ |
|---|---|---|---|
| Isentropic Flow | area change | constant | constant |
| Fanno Flow | friction | **not** constant | constant |
| Rayleigh Flow | heating/cooling | **not** constant | **not** constant |

This table provides a "road map" for our study of frictional and heat transfer flow in this and the next chapter.

Friction, of course, is present in all fluid flow.* In developing isentropic flow, we assumed that the friction effects are small, or are confined to a thin layer near the wall (i.e., the boundary layer) outside of which the flow is very nearly inviscid. This approximation is valid for internal flows if the channel is short (i.e., only a few channel diameters long), such as the nozzles and diffusers discussed in Sections 6.3 and 9.2. If the channel is long, however, then the effects of friction can become the dominant influence on the flow and clearly cannot be neglected. By a "long" channel, we are referring to a channel that is many characteristic diameters long. Obvious examples of this type of flow occur in pipelines, although frictional flow is used to model a number of other flows that appear in a variety of applications.

In this chapter, we will consider the effect of friction in constant area channels only. Similar to our development of isentropic flow, we will assume fluid properties are uniform across the cross sectional area of the channel, so the flow is assumed one dimensional. This may seem an assumption that is difficult to justify, since we know that viscosity will manifest itself in phenomenon such as shear, boundary layers, turbulence, etc., all of which are inherently multi-dimensional. Nonetheless, we will treat the effect of friction as being a force that is applied uniformly to the entire cross sectional area of the flow, and assume the flow responds such that the flow properties only vary along the *x*-direction of the channel. Despite the gross oversimplification of this treatment, we will see in this chapter that the resulting relations still manage to "capture" many of the aspects of real frictional flow. The resulting tools are useful in analyzing a variety of real engineering problems.

Our treatment will also assume there is no heat transfer to the flow from the channel walls, i.e., adiabatic flow. This again may seem to be an assumption that is not consistent with a flow in which friction is a significant factor. After all, both friction and heat transfer are dominated by molecular transport, how can you have one without the other? In fact, the quantitative equivalence of heat transfer and friction [Prandtl's relation] is an important relation in heat transfer. Nonetheless, it is of interest to examine the effect of friction alone,

*Again, there are exceptions, if you look hard enough. The phenomenon of "superfluidity," which occurs in liquid helium-4 at supercooled temperature ($T < 2$ K), results in fluid motion with apparently no viscosity. This is attributed to a quantum mechanical effect that is still not completely understood, similar to superconductivity.

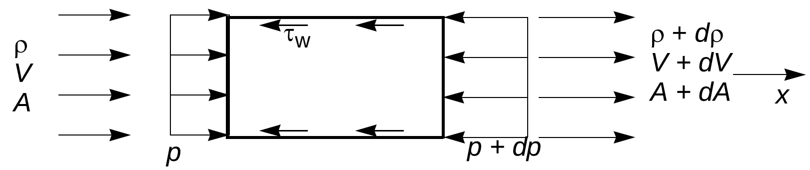
without heat transfer. This permits analytic solutions of the governing equations, which is not in general possible with the combined effect of heat transfer and friction.

Similar to isentropic flow, we will examine *steady flows* (flows that do not change with time) or *quasi-steady flows* (flows that change so slowly, they can be treated as a sequence of separate, steady flows).

The governing equations for steady, one-dimensional adiabatic flow are (from Section 3.3):

| | Control Volume | Differential |
|---|---|---|
| Continuity | $\rho_1 \boldsymbol{V}_1 A_1 = \rho_2 \boldsymbol{V}_2 A_2$ | $\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$ |
| Momentum | $F_{xwall} = A_2[p_2 + \rho_2 V_2^2] - A_1[p_1 + \rho_1 V_1^2]$ <br> $= A_2 p_2 + V_2 \dot{m} - A_1 p_1 - V_1 \dot{m}$ | |
| Energy | $\left(h + \frac{V^2}{2}\right) = \text{constant}$ | $dh + VdV = 0$ |

Note that the continuity equation and energy equations developed in Chapter 3 for steady, one-dimensional adiabatic streamtubes apply directly. For the momentum equation, the same control volume (or integral) relation still applies, except now we must be aware that the $F_{xwall}$ term includes the force of wall friction on the fluid. The differential version of the momentum equation developed in Chapter 3, however, cannot be used, which is why it is left blank in the table above. Recall, in developing this relation in Section 3.2, we assumed that in an inviscid flow, the only wall force is pressure. Thus, it is necessary to re-derive the differential momentum equation taking wall friction into account.




Here, $\tau_w$ is the shear stress [N/m$^2$] at the wall due to the viscosity of the fluid. The total frictional force acting on the fluid inside this different element of flow is: $F_{xwall} = \tau_w A_{wet}$ in the $-x$ direction. $A_{wet}$ is the wetted area, i.e., the surface area inside the channel that would get wet if we flowed water through the channel. Applying the momentum equation for this constant area ($A_1 = A_2$) control volume:

$$A_1 p_1 + V_1 \dot{m} + F_{xwall} = A_2 p_2 + V_2 \dot{m}$$

$$Ap + V\dot{m} - \tau_w A_{wet} = A(p + dp) + (V + dV)\dot{m}$$

Canceling out higher order terms:

$$dp + \rho V dV + \tau_w \frac{A_{wet}}{A} = 0$$

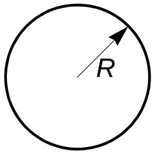
Shear stress $\tau_w$ usually scales with $\frac{1}{2}\rho V^2$, so it is convenient to define a friction factor $f = \dfrac{\tau_w}{\frac{1}{2}\rho V^2}$. The value of $f$ is, in general, a complex function of the flow Reynolds number, Mach number, and the surface conditions at the wall (roughness, etc.). The values of $f$ for different regimes of flow can be represented on the well known "Moody chart." Typically, for aerospace applications where Reynolds number is large (Re > $10^6$), $f$ usually has a value in the range $0.001 < f < 0.05$. In this chapter, we will typically assume a fixed value of $f$ in this range. In actual applications of one-dimensional frictional flow, a common practice is to assume two different values of $f$ near the limits of this range, to provide a "best case" and "worst case" bound on the effects of friction on a flow.

The wetted area $A_{wet}$ is related to the internal perimeter $P$ of the channel by $A_{wet} = P\,dx$. The perimeter, in turn, is related to the *hydraulic diameter* by $D_H = \frac{4A}{P}$. The hydraulic diameter may seem an unusual parameter, but examining a few basic shapes of the cross sectional channel reveals that it is simply a representative diameter for the channel. For example:

<u>Circular Cross Section</u>

Area: $A = \pi R^2$

Perimeter: $P = 2\pi R$



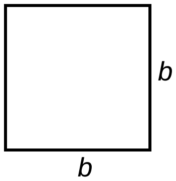

Hydraulic Diameter: $D_H = \frac{4A}{P} = \frac{4(\pi R^2)}{2\pi R} = 2R$

Thus, the hydraulic diameter of a circular pipe is simply the diameter of the pipe.

<u>Square Cross Section</u>

Area: $A = b^2$

Perimeter: $P = 4b$




Hydraulic Diameter: $D_H = \frac{4A}{P} = \frac{4b^2}{4b} = b$

Thus, the hydraulic diameter of a square pipe is simply the width of one side.

For more complex shaped cross sections, the hydraulic diameter becomes a convenient representative dimension for the purposes of determining the force of friction.

Using the friction factor $f$ and the hydraulic diameter $D_H$, the differential momentum equation becomes:

$$dp + \frac{4f}{D_H}\frac{\rho V^2}{2}dx + \rho V dV = 0 \tag{10.1}$$

Note that when friction is zero ($f = 0$), this equation reverts to our earlier inviscid momentum equation. We now have a complete "toolbox", ready for use with one-dimensional, steady, constant area flow with friction.

## 10.1 Effect of Friction

The first question we would like to answer is: What effect does friction have on a flow? Does it accelerate or decelerate the flow? Does pressure, temperature, density, etc. increase or decrease under the influence of friction? Intuitively, you might expect the effect of friction to decelerate a flow. Like isentropic flow, however, frictional flow has some surprises that challenge our intuition.

Recall in developing isentropic flow (Chapter 5), the first step we took was to find a differential relation relating a change in area to a change in velocity $\left(\frac{dV}{V} = \frac{1}{(M^2-1)}\frac{dA}{A}\right)$. This relation was fundamental to understanding how isentropic flow responds to area change (i.e., subsonic flow decelerates in a diverging area, supersonic flow accelerates, etc.). We would like to develop a similar relation to see how flow responds to a section of channel with friction.

Following the same procedure we did in Chapter 5 for isentropic flow, we will begin with the differential momentum equation:

$$dp + \frac{4f}{D_H}\frac{\rho V^2}{2}dx + \rho V dV = 0$$

and eliminate all the differentials except the one were are interested in. We will retain the $\frac{4f}{D_H}dx$ term, since this is the factor influencing the flow, and isolate the other parameter we are interested in. To begin with, we will try to see how velocity change $dV$ responds to friction. To eliminate density $\rho$, we will divide through by $p$:

$$\frac{dp}{p} + \frac{4f}{D_H}\frac{1}{2}\frac{\rho V^2}{p}dx + \frac{\rho V}{p}dV = 0$$

and use the now familiar substitution $\rho V^2 = \gamma p M^2$, leaving:

$$\frac{dp}{p} + \frac{4f}{D_H}\frac{1}{2}\gamma M^2 dx + \gamma M^2\frac{dV}{V} = 0 \qquad (10.2)$$

To eliminate $\frac{dp}{p}$, we can use the differential form of the equation of state ($p = \rho RT$):

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Since the flow is constant area ($\rho V$ = constant), we can replace $\frac{d\rho}{\rho} = -\frac{dV}{V}$:

$$\frac{dp}{p} = -\frac{dV}{V} + \frac{dT}{T} \qquad (10.3)$$

We have eliminated $\frac{dp}{p}$, but now we have a $\frac{dT}{T}$ term. To express this in terms of velocity, we can use the differential energy equation:

$$dh + VdV = 0$$

$$c_p dT + V^2\frac{dV}{V} = 0$$

Dividing through by $c_p T$ and using $c_p = \frac{\gamma R}{\gamma - 1}$:

$$\frac{dT}{T} = -(\gamma - 1)M^2\frac{dV}{V} \qquad (10.4)$$

Substituting (10.4) into (10.3), and (10.3) in turn into (10.2), we obtain:

$$\boxed{\frac{dV}{V} = \frac{\gamma M^2}{2(1-M^2)}\frac{4f}{D_H}dx} \qquad (10.5)$$

This was our objective: a differential relation relating the change in velocity $dV$ to the effect of friction acting over a section of channel $dx$. We will explore the implications of this relation in a moment, but first we can obtain a similar expression for $\frac{dM}{M}$ by starting with the definition of Mach number ($M^2 = \frac{V^2}{\gamma RT}$) and applying logarithmic differentiation:

$$\frac{dM}{M} - \frac{dV}{V} = -\frac{1}{2}\frac{dT}{T}$$

Combining this with (10.4) gives

$$\frac{dM}{M} = \left(1 + \frac{\gamma - 1}{2}M^2\right)\frac{dV}{V}$$

So, (10.5) can be converted into an expression for the change in Mach number $dM$:

$$\boxed{\frac{dM}{M} = \frac{\gamma M^2\left(1 + \frac{\gamma-1}{2}M^2\right)}{2(1-M^2)}\frac{4f}{D_H}dx} \tag{10.6}$$

These relations for $\frac{dV}{V}$ and $\frac{dM}{M}$ obey a similar trend:

- For *subsonic flow* ($M < 1$), the effect of friction is to increase velocity and Mach number ($dV > 0$, $dM > 0$). *The flow accelerates*.
- For *supersonic flow* ($M > 1$), the effect of friction is to decrease velocity and Mach number ($dV < 0$, $dM < 0$). *The flow decelerates*.

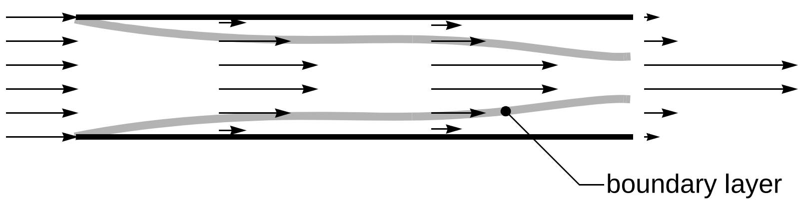
We can continue to manipulate these differentials and isolate the variables of interest so that their differential is only a function of Mach number and the friction term $\frac{4f}{D_H}dx$. We will summarize these results here:

$$\frac{dT}{T} = -\frac{\gamma(\gamma-1)M^4}{2(1-M^2)}\frac{4f}{D_H}dx \tag{10.8}$$

$$\frac{dp}{p} = -\frac{\gamma M^2[1+(\gamma-1)M^2]}{2(1-M^2)}\frac{4f}{D_H}dx \tag{10.9}$$

We notice that the effect of friction is qualitatively the same as converging area: in *subsonic flow*, it results in accelerating flow, with a decrease in static pressure and temperature. In *supersonic flow*, it results in decelerating flow, and an increase in static pressure and temperature. Like a converging area, friction always acts to drive a flow toward Mach 1.

The comparison between the effect of friction and area convergence is not just a coincidence. If we examine a *real* friction flow, we would observe the presence of boundary layers developing in the channel, which have the effect of generating a *converging area profile* for the core flow, resulting in its acceleration in the case of a subsonic flow:




Thus, the effect of friction is indeed similar to that of converging area. It is remarkable that our one-dimensional treatment of frictional flow effectively "captures" this feature of real, multidimensional flow with boundary layers.*
If you can remember this picture (that a constant area friction flow behaves similar to a converging isentropic flow), you should have an intuitive feel for how friction will effect the flow.

*It is sometimes said that the test of a good theory or model is its successful application outside of the narrow set of assumptions used in its derivation. The Fanno flow relations certainly meet this test! When you start with the basic conservation laws and apply them consistently, you are often rewarded with a tool that works better than it should!

It is important to point out how frictional flow is different than isentropic flow. As mentioned before, the presence of friction will results in irreversibility, so entropy will increase through the flow and stagnation pressure will decrease.

Comparison of Frictional Flow (Converging Area Isentropic Flow)
for Subsonic and Supersonic Flow

| Property | Subsonic | Supersonic |
|---|---|---|
| $M$ | increases | decreases |
| $V$ | increases | decreases |
| $p$ | decreases | increases |
| $T$ | decreases | increases |
| $p_o$ | decreases (constant) | decreases (constant) |
| $T_o$ | constant | constant |

Note that, other than stagnation pressure, the variation of properties in frictional flow is in the same direction as converging isentropic flow.

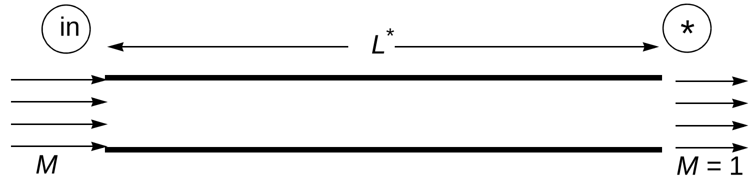
## 10.2 Working Relations for Frictional Flow

Since we have seen that friction, like area convergence, drives a flow toward sonic, it is convenient to use the conditions where the flow reaches sonic as a reference state, similar to how we used stagnation conditions and sonic conditions isentropic flow.

Of particular interest is the length of channel required for the effects of friction to drive the flow to sonic, which we will denote $L^*$. Starting with the different relation:

$$\frac{dM}{M} = \frac{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{2(1-M^2)}\frac{4f}{D_H}dx$$

we can integrate from the inlet conditions M to the sonic condition at the end of the pipe, ($M = 1$) at the end of the channel of length $L^*$.




$$\int_0^{L^*}\frac{4f}{D_H}dx = \int_M^1 \frac{2(1-M^2)}{M^2\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

The right hand side can be integrated by the method of partial fraction expansions to yield:

$$\boxed{\frac{4fL^*}{D_H} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)}} \qquad (10.10)$$

This relation tells us the length $L^*$ required to bring a flow entering a channel at Mach number $M$ to sonic. Note that the $\frac{4fL^*}{D_H}$ term is really just a way to non-dimensionalize $L^*$. This relation will play a role similar to the one played by the sonic area relation $\left(\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right)$ for isentropic flow. For a given flow condition at a point in the flow, we can use the known information to determine $L^*$, similar to how we determine sonic area $A^*$ in isentropic flow. We can then use this state to solve for other points in the same flow, even if it does not actually reach sonic conditions due to friction.

Likewise, we can integrate relations (10.8) and (10.9) for temperature and pressure, respectively:

$$\int_T^{T^*} \frac{dT}{T} = \int_M^1 \frac{-(\gamma-1)M\,dM}{1+\frac{\gamma-1}{2}M^2} \tag{10.11}$$

$$\frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M^2} \tag{10.12}$$

Note that we could have obtained this relation directly from the energy equation for adiabatic flow:

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$T_1\left(1+\frac{\gamma-1}{2}M_1^2\right) = T_2\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

Taking $T_1 = T$, $M_1 = M$, and $T_2 = T^*$, $M_2 = 1$:

$$\frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M^2} \Leftarrow \text{same as (10.12)}$$

Integrating for pressure and stagnation pressure, we obtain:

$$\frac{p}{p^*} = \frac{1}{M}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}} \tag{10.13}$$

We can obtain a similar expression for stagnation pressure:

$$\frac{p_o}{p_o^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{10.14}$$

The values of $\frac{T}{T^*}$, $\frac{p}{p^*}$, $\frac{p_o}{p_o^*}$, and $\frac{4fL^*}{D_H}$ are tabulated as a function of Mach number in Table A-4.

To illustrate the application of these relations, we will consider a numerical example.

### 10.2.1 Numerical Example: Subsonic Flow in a Channel with Friction

**Problem:** Consider air flowing into a pipe at Mach 0.5 and static pressure and temperature of 1 atm and 300 K, respectively. The channel has a diameter of 5 cm and a coefficient of friction $f = 0.005$.

(a) Find the length of channel $L^*$ required for the flow to reach sonic. Find $p^*$, $T^*$, and $p_o^*$.

(b) If the pipe is $L = 1.5$ m long, find the Mach number and conditions $p$, $T$, and $p_o$ at the exit.

**Solution:**

**(a) Sonic Exit Flow**

We begin by finding the reference conditions from the information we are given. Since we are given static pressure and Mach number, we can find the stagnation conditions at the entrance to the channel, which we will denote with subscript “1”:

$$p_{o1} = p_1\left(\frac{p_o}{p}\right)_{M=0.5} = (1\text{ atm})(1.186) = 1.186\text{ atm}$$

Note that, even though we are dealing with a frictional flow, we continue to use isentropic relations to find the stagnation conditions for a given point in the flow. We must recall, however, that stagnation pressure will *not* be a constant reference state as flow passes though the channel.

Since we know the entrance Mach number into the pipe, we can use Equation 10.10 (or its tabulated values in Table A-4) to find the length of channel required to bring this flow to sonic:

$$\left(\frac{4fL^*}{D_H}\right)_{M\,=\,0.5} = 1.069$$

Since we know the friction coefficient $f$ and the hydraulic diameter $D_H$, we can find $L^*$:

$$L^* = 1.069\left(\frac{D_H}{4f}\right) = 1.069\left(\frac{0.05\text{ m}}{4\cdot 0.005}\right) = 2.67\text{ m}$$

Thus, if the pipe is $L^*$ = 2.67 m long, the flow at the exit will be sonic. The conditions at the exit will simply be the sonic conditions:

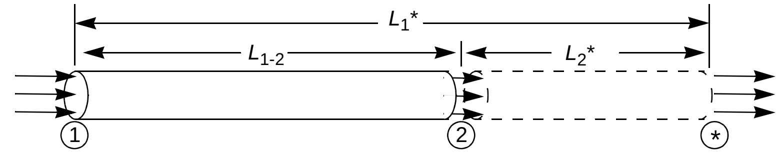
$$\left(\frac{T}{T^*}\right)_{M\,=\,0.5} = 1.1429 \Rightarrow T^* = \frac{300\text{ K}}{1.1429} = 262.5\text{ K}$$

$$\left(\frac{p}{p^*}\right)_{M\,=\,0.5} = 2.1381 \Rightarrow p^* = \frac{1\text{ atm}}{2.1381} = 0.468\text{ atm}$$

$$\left(\frac{p_o}{p_o^*}\right)_{M\,=\,0.5} = 1.3399 \Rightarrow p_o^* = \frac{1.186\text{ atm}}{1.3399} = 0.885\text{ atm}$$

**(b) Exit Flow with L = 1.5 m**

In this case, the pipe is shorter than $L^*$, so we expect the exit flow to be subsonic. How will we find the exit conditions? The solution technique is to consider that this is the same channel flow as in (a), only we have cut the channel off at location 2 ($L$ = 1.5 m), before it has reached sonic at ⊛. If we added an extension of channel $L_2^* = L_1^* - L_{1\text{-}2}$ = 2.67 m – 1.5 m = 1.17 m back onto the channel, we should obtain the same exit flow as in part (a). The picture then is:




The flow exiting the first channel ($L_{1\text{-}2}$ = 1.5 m) must be the same flow entering the second, imaginary channel we have added on ($L_2^*$ = 1.17 m). Since the flow entering this second channel will be sonic at the exit, we know the value of $\frac{4fL_2^*}{D_H}$, which can be related to the entrance Mach number (which is also the *exit* Mach number of the original channel):

$$\frac{4fL_2^*}{D_H} = \frac{4\cdot 0.005\cdot 1.17\text{ m}}{0.05\text{ m}} = 0.468 \Rightarrow M_2 = 0.61$$

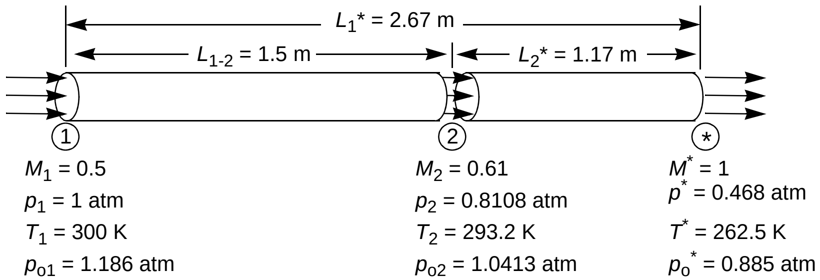
Once we know the Mach number, we can relate the other properties to the sonic properties we found in part (a) using our frictional flow relations:

$$\left(\frac{T}{T^*}\right)_{M\,=\,0.61} = 1.1169 \Rightarrow T_2 = 1.1169\cdot 262.5\text{ K} = 293.2\text{ K}$$

$$\left(\frac{p}{p^*}\right)_{M\,=\,0.61} = 1.7325 \Rightarrow p_2 = 1.7325\cdot 0.468\text{ atm} = 0.8108\text{ atm}$$

$$\left(\frac{p_o}{p_o^*}\right)_{M\,=\,0.61} = 1.1766 \Rightarrow p_{o2} = 1.1766\cdot 0.885\text{ atm} = 1.0413\text{ atm}$$

As always, it is a good idea to construct a summary of our results:




It is important to study these results so that we develop an intuitive feel for frictional flow, similar to what we did for isentropic flow. As the flow traversed the frictional channel, it accelerated. This result may seem counterintuitive, but if we examine the static pressure, it decreased from 1 atm at the inlet to 0.468 atm at the sonic end. Thus, there is a large pressure gradient that is accelerating the gas as it responds to the effect of friction. Because the flow accelerates, static temperature decreases. And, unlike isentropic flow, the effect of friction has resulted in an irreversible decrease in the stagnation pressure $p_o$.

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This numerical example raises a number of questions. What if we had only been asked part (b) in the above problem: given an inlet flow, what is the exit flow after a given length of channel (say, $L$ = 1.5 m). Then, in order to solve for the conditions at location 2, we would still have had to find the length required to reach sonic conditions and what those conditions are ($p^*$, $T^*$, etc.). Thus, we would have had to do part (a) of the problem anyway! This is similar to isentropic flow where we often used reference states (like sonic area $A^*$ and stagnation pressure $p_o$) even though the flow may never have reached sonic or stagnated in that particular problem.

Note that it *is* possible to solve directly for the conditions at location 2 in terms of the known conditions at location 1. If we return to the differential relation for Mach number (Eq 10.6):

$$\frac{dM}{M} = \frac{\gamma M^2\left(1 + \frac{\gamma-1}{2}M^2\right)}{2(1-M^2)}\frac{4f}{D_H}dx$$

We can integrate from an arbitrary inlet Mach number $M_1$ to an arbitrary exit Mach number $M_2$ (rather than to sonic):

$$\int_0^{L_{1-2}} \frac{4f}{D_H}dx = \int_{M_1}^{M_2} \frac{2(1-M^2)}{M^2\left(1 + \frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

This can be integrated to yield:

$$\frac{4fL_{1-2}}{D_H} = \frac{1}{\gamma}\left(\frac{1}{M_1^2} - \frac{1}{M_2^2}\right) + \frac{\gamma+1}{2\gamma}\ln\frac{M_1^2}{M_2^2}\left(\frac{1 + \frac{\gamma-1}{2}M_2^2}{1 + \frac{\gamma-1}{2}M_1^2}\right) \tag{10.15}$$

This relation directly relates the Mach number at location 1 to the Mach number at location 2; we can derive similar relations relating $p_1$ and $p_2$, $T_1$ and $T_2$, etc. Note, however, that given $L_{1-2}$ and $M_1$ (as we were in part (b) of the numerical example above), we cannot analytically solve (10.15) for $M_2$, and we would likely have to guess and iterate on a solution. We could compile tables of $M_1$, $M_2$, and $\frac{4fL^*}{D_H}$, however these tables would be huge since we have two independent variables. Thus, it ends up being a more efficient solution technique to relate the given flow properties to the sonic condition that would be obtained if the pipe was long enough, and then “backtrack” to find the flow at the location we are interested in, as we did in the numerical example above. With modern programmable calculators and computers, however, it is possible to solve directly for the flow at a new location by using a root-finding algorithm for Eq. 10.15 and the corresponding relations for pressure, temperature, etc.

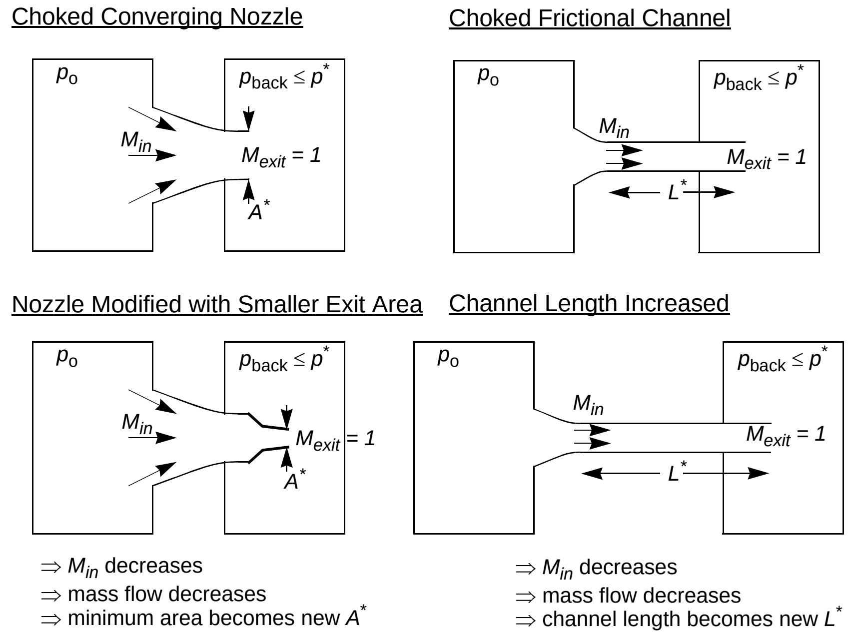
Another obvious question raised by this example is: what would happen if the channel were made longer than $L^*$? This question is dealt with in the next section.

## 10.3 Effect of Channel Length

The effect of friction is to always drive a flow toward sonic conditions, reaching sonic at a distance $L^*$ from the entrance. What happens if the channel is made longer than $L^*$? The answer to this question can be best understood by considering the analogous question from isentropic flow: What happens if a channel is made smaller in area than $A^*$? This is a question that was discussed in some detail in Chapters 6 and 9. The answer depends on the type of flow (subsonic *vs*. supersonic).

For a *subsonic isentropic flow*, such as the flow generated by a converging nozzle fed by a quiescent reservoir, consider the case where the flow discharges into a chamber with a back pressure $p_{back} \le p^*$. In this case, the flow will be choked (sonic) at the exit of the converging nozzle at area $A^*$. If an extension is added to the nozzle that further converges the area, then the flow will adjust itself so that the new minimum area is now sonic. Since the choke point is now a smaller area, the mass flow rate $\dot{m}$ through the channel will decrease. The flow can adjust because, being subsonic, the fact that a smaller area exists can be communicated upstream.

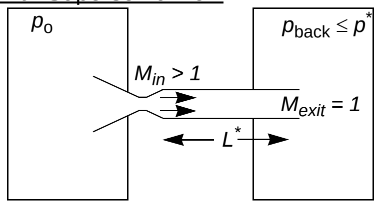
The analogous case occurs for a frictional channel with subsonic inflow. If the channel length is increased beyond $L^*$ while $p_o$ and $p_{back}$ remain constant, the inlet conditions at the channel inlet *must* adjust themselves so that the new length is the sonic length. This will result in a lower inlet Mach number (i.e., the flow starts further from sonic, so that it must flow for a longer distance before reaching sonic) and a lower mass flow rate $\dot{m}$. When a frictional channel reaches sonic conditions, we say that it is *choked* by the effect of friction because increasing the length of channel necessitates a decrease in the mass flow rate, just as an isentropic flow can be choked by a minimum in area.



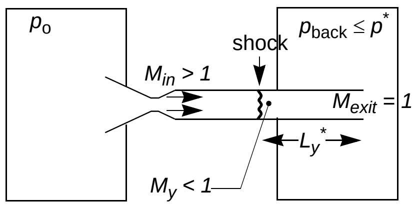

For *supersonic* flow in a frictional channel, the back pressure must also be $p_{back} \le p^*$ in order for the exit flow to remain supersonic. If the channel length is increased beyond $L^*$, the fact that the flow is approaching a channel that is too long to accept the mass flow cannot be communicated upstream in a supersonic flow. Thus, a *shock wave* is forced upstream. This is analogous to the "unstart" phenomenon we encountered with supersonic diffusers and windtunnels (Sections 9.2-9.3). Unlike in isentropic flow, a normal shock *is* stable in a constant area frictional channel. The shock becomes a "patch" between supersonic channel flow upstream and subsonic flow downstream. The flow at the exit of the channel is still choked in this case. This is due to the fact that, for supersonic flow entering a constant area frictional channel, $p^*$ is *constant across a normal shock*.* Thus, since the back pressure is $p_{back} \le p^*$, the flow will continue to choke at the exit.

*This property of Fanno flow will become useful in problem solving.

Choked Frictional Channel
with Supersonic Flow




Channel Length Increased




⇒ Normal shock driven into channel
⇒ mass flow constant
⇒ channel length becomes new $L^*$

The normal shock will move further upstream as the flow is further constricted downstream by increasing channel length. The shock can be forced up into the nozzle and, eventually, to the throat. In this case, further increase in the channel will result in subsonic flow throughout, and the problem becomes the case of subsonic flow discussed above: further increase in channel length will result in a reduction in mass flow rate.

One question that may occur in the case of the shock existing in a frictional channel: how do we know that the flow will not choke in the subsonic flow downstream of the normal shock before reaching the end of the channel? The answer is that subsonic flows take considerably longer to choke than supersonic flow. We can see this by simply scanning the $\frac{4fL^*}{D_H}$ column in the Fanno Flow table (Table A-4). Note that, for subsonic Mach numbers, $\frac{4fL^*}{D_H}$ increases rapidly as we move away from Mach 1; it takes a longer and longer distance to choke the flow. For supersonic flow, $\frac{4fL^*}{D_H}$ increases slowly as we move away from Mach 1 and asymptotes to a finite value as $M \to \infty$:

$$\lim_{M \to \infty} \frac{4fL^*}{D_H} = \lim_{M \to \infty}\left(\frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right)$$

$$\lim_{M \to \infty} \frac{4fL^*}{D_H} = \frac{-1}{\gamma} + \frac{\gamma+1}{2\gamma}\ln\frac{\gamma+1}{\gamma-1}$$

For typical values of $f$ ($f = 0.002$, $\gamma = 1.4$), $\frac{L^*}{D_H} \approx 100$. Thus, even with very high Mach number flow entering the channel, friction requires less than one hundred channel diameters to choke the flow. For subsonic flow with similar values of $f$ and $\gamma$, flows of Mach number less than $M = 0.5$ require more than 100 channel diameters to choke. Thus, it can be shown that the subsonic flow following a normal shock will always require a longer channel choke than the original supersonic flow. When the channel length is increased beyond $L^*$ for the supersonic flow in a frictional channel, the normal shock positions itself so that the combination of supersonic and subsonic frictional flow results in a flow that is still choked at the exit.

In this section, we assumed that the stagnation pressure $p_o$ and back pressure $p_{back}$ were fixed. How the flow responds to varying back pressure is discussed in the next section.

## 10.4 Effect of Back Pressure

In addition to varying the channel length, the flow in a frictional channel can be controlled by varying back pressure. For a channel with *subsonic* flow entering the channel (e.g., as fed by a converging nozzle), decreasing the back pressure increases the inlet Mach number and mass flow rate until sonic conditions are reached at the exit of the channel ($p_{back} = p^*$). Further decrease in back pressure does not affect the flow in the channel.

For a channel with *supersonic* flow at the inlet (e.g., channel fed by a converging-diverging nozzle), the details of the flow depend on if $L$ is longer or shorter than $L^*$ as defined by the supersonic inlet condition. If $L < L^*$, then the flow may remain entirely supersonic if the back pressure is close enough to the exit pressure determined by the supersonic solution for Fanno flow. As the back pressure is increased, a normal shock will form at the exit of the channel. As the back pressure is increased further, the shock will move further upstream. The exit flow will remain subsonic in this case; if the channel was not long enough to choke the supersonic inflow, it cannot choke the subsonic flow downstream of the normal shock. Once a shock appears, the exit pressure must match the back pressure. If the back pressure increases sufficiently to force the shock into the diverging nozzle and to the throat, the shock will disappear and the flow will be subsonic throughout. Further increases in back pressure will reduce the mass flow rate.

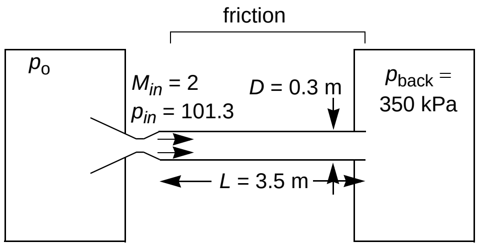
For the case where the channel is longer than $L^*$ as defined by the supersonic inlet conditions, a shock wave *must* occur. The shock wave in this case cannot exist at the exit of the channel (that would require supersonic flow to exceed $L^*$); the shock must occur upstream from the exit. If the back pressure is less than $p^*$ as defined by the supersonic inlet conditions (recall $p^*$ is constant across the shock), the exit flow will be sonic and need not match the back pressure. If $p_{back} > p^*$, then the shock will be forced further upstream and the exit flow will be subsonic. Increasing the back pressure further forces the shock further upstream until it reaches the throat, establishing subsonic flow throughout.

In solving a problem in which the channel length, stagnation pressure, and back pressure are given, it is often difficult to *a priori* decide which flow condition exists (presence of shock in channel, supersonic/sonic/subsonic exit flow, etc.). In order to determine the flow, a good procedure is to simply *assume* a flow condition (e.g., supersonic flow throughout) and then "march" thought the flow and determine the exit pressure. Comparing this exit pressure to the specified back pressure will indicate the possible flow conditions that may or may not exist. For example, if the back pressure is below the exit pressure for the case of assuming entirely supersonic flow, then supersonic flow can exist throughout and expansions will occur beyond the end of the channel. If the back pressure is greater than the computed exit pressure for the case of supersonic flow, then a normal shock may exist in the channel. Repeating the calculation with a normal shock at the entrance of the channel will provide another value of exit pressure which can be compared to the back pressure. This procedure will quickly narrow in on the correct flow solution.

This procedure is best illustrated by a numerical example.

### 10.4.1 Numerical Example: Supersonic Flow in Frictional Channel when $L < L^*$

**Problem:** Flow enters a channel with friction from a converging-diverging nozzle at Mach 2.0 and static pressure of 101.3 kPa. The channel is 3.5 m long, 30 cm in diameter, and has a coefficient of friction $f = 0.005$. The channel exhausts into a chamber with back pressure $p_{back} = 350$ kPa. Find the exit Mach number.




**Solution**: As always, we will define our reference states. As we are told the Mach number and static pressure of the flow entering the channel, we can determine the stagnation pressure (i.e., the pressure in the reservoir that is feeding the channel):

$$\left(\frac{p}{p_o}\right)_{M=2} = 0.12780 \Rightarrow p_o = 792.6 \text{ kPa}$$

$$\left(\frac{A}{A^*}\right)_{M=2} = 1.6875$$

It is not clear whether, for the given back pressure, a normal shock exists in the channel or not. Thus, we will begin by assuming that there is no shock, and see what the exit pressure is in this case:

(a) Supersonic Flow in Channel

For an inflow of $M_{in} = 2$, the length of channel required to reach sonic is:

$$\left(\frac{4fL^*}{D_H}\right)_{M=2} = 0.30499 \Rightarrow L^* = 4.575 \text{ m}$$

Note that if the channel were this long and sonic conditions exited at the exit, then the exit pressure would be:

$$\left(\frac{p}{p^*}\right)_{M=2} = 0.40825 \Rightarrow p^* = 248.1 \text{ kPa}$$

The channel we are given ($L = 3.5$ m) is less than $L^*$: $L < L^*$. There may or may not be a shock in this case; we must examine the back pressure before deciding. The flow exiting the channel in the case of supersonic flow must be the same flow that would enter a channel of length $L_2^*$:

$$L_2^* = L_1^* - L_{1-2} = 4.575 - 3.5 = 1.075 \text{ m}$$

(Review Numerical Example 10.2.1 for a further explanation of this procedure). Using the Fanno Flow relations or Table A-3:

$$\frac{4fL_2^*}{D_H} = \frac{4 \cdot 0.005 \cdot 1.075}{0.30} = 0.07166 \Rightarrow M_{exit} = 1.32$$

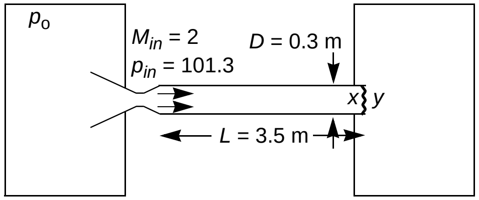
Since the sonic pressure we found above applies to the entire flow, the exit pressure in this case must be:

$$\left(\frac{p}{p^*}\right)_{M=1.32} = 0.71465 \Rightarrow p_{exit} = 177.3 \text{ kPa}$$

This exit pressure is significantly less than the actual back pressure. Thus, we can conclude that their may be a normal shock in the channel.

(b) Normal Shock at Channel Exit

For our solution for the case of supersonic flow though the channel (a), it is easy to apply the normal shock conditions to the exit flow:



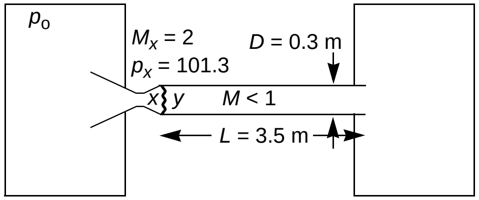

$$M_x = 1.32 \Rightarrow M_y = 0.776$$

$$M_x = 1.32 \Rightarrow \frac{p_y}{p_x} = 1.866 \Rightarrow p_y = 330.8 \text{ kPa}$$

The exit pressure in this case does not match the back pressure, so this is not the correct solution (note that, since the flow is subsonic, the exit pressure would have to match the back pressure). The exit pressure is closer to the given back pressure, however, so we are getting closer the correct solution. To increase the exit pressure, we will move the shock further upstream.

(c) Normal Shock at Channel Entrance

To determine if the shock is in the channel, we now position the shock at the other extreme: at the entrance of the channel.



Since we know the flow conditions there ($M_{in}$ = 2, $p_{in}$ = 101.3 kPa):

$$(M_y)_{M_x = 2.0} = 0.5774$$

$$\left(\frac{p_y}{p_x}\right)_{M_x = 2.0} = 4.500 \Rightarrow p_y = 455.85 \text{ kPa}$$

We can now solve the problem of the subsonic flow downstream of the shock in a frictional channel 3.5 m long:

$$\left(\frac{4fL^*}{D_H}\right)_{M = 0.5774} = 0.5898 \Rightarrow L^* = 8.847 \text{ m}$$

$$\left(\frac{p}{p^*}\right)_{M = 0.5774} = 1.837 \Rightarrow p^* = 248.1 \text{ kPa}$$

Note that the sonic pressure for the subsonic flow following the normal shock ($p^*$ = 248.1 kPa) is identical to the sonic pressure we found in (a) where the flow was supersonic. This is because, as mentioned previously, $p^*$ *is constant across a shock in a channel*. For a given inflow condition (such as in this problem), the same sonic pressure reference state applies regardless of the location or presence of a shock in the channel. Thus, it was unnecessary to compute it again here. The sonic length $L^*$ *does* change as the flow transitions from supersonic to subsonic via the normal shock. Knowing this length, we can find the exit conditions:

$$\frac{4fL_2^*}{D_H} = \frac{4fL_1^*}{D_H} - \frac{4fL_{1-2}}{D_H} = 0.5898 - \frac{4 \cdot 0.005 \cdot 3.5}{0.3} = 0.3565$$

(Note that it is convenient to calculate using the parameter $\frac{4fL}{D_H}$, since this is the parameter tabulated in Table A-4, rather than using the physical length in meters.)

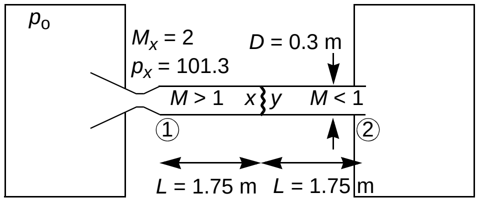
$$(M_2)_{\frac{4fL_2^*}{D_H} = 0.3565} = 0.639$$

$$\left(\frac{p_2}{p^*}\right)_{\frac{4fL_2^*}{D_H} = 0.3565} = 1.649 \Rightarrow p_2 = 408 \text{ kPa}$$

Thus, the exit pressure in this case must be $p_{exit}$ = 408, which is *greater* than the back pressure. We have therefore bounded the location of the normal shock: it must be located in the channel.

(d) Shock at Halfway Point

Since we know the shock is in the channel, we can position the shock at the halfway point to further narrow in on its location:




Solving the supersonic frictional flow through the first 1.75 m of channel:

$$\frac{4fL_x^*}{D_H} = \frac{4fL_1^*}{D_H} - \frac{4fL_{1-x}}{D_H} = 0.30499 - \frac{4 \cdot 0.005 \cdot 1.75}{0.3} = 0.1883$$

$$(M_x)_{\frac{4fL_2^*}{D_H} = 0.1883} = 1.645$$

Note that the presence of the shock does not influence our solution for the conditions "$x$" immediately upstream of the shock. Since the flow is supersonic, we use the sonic length $L^*$ = 4.575 m as if the flow were going to continue supersonic. Once we have found the conditions at "$x$", we can apply the normal shock relations to find "$y$":

$$M_x = 1.645 \Rightarrow M_y = 0.6550$$

We now solve a new frictional flow: subsonic flow entering a 1.75 m long channel at Mach number $M_y = 0.6550$. We have a new sonic length $L_y^*$ for this case, but the sonic pressure $p^*$ remains the same:

$$\frac{4fL_2^*}{D_H} = \left(\frac{4fL_y^*}{D_H}\right)_{M_y = 0.655} - \frac{4fL_{y-2}}{D_H} = 0.3112 - \frac{4 \cdot 0.005 \cdot 1.75}{0.3} = 0.1945$$

$$(M_{exit})_{\frac{4fL_2^*}{D_H} = 0.1945} = 0.705$$

$$\left(\frac{p_2}{p^*}\right)_{\frac{4fL_2^*}{D_H} = 0.1945} = 1.482 \Rightarrow p_2 = 367.7 \text{ kPa}$$

Thus, $p_{exit} = 367.7$ kPa for the case of the shock at the halfway point, which is nearer to the actual back pressure of $p_{back} = 350$ kPa. Thus, we can interpolate between this solution and the solution from (c) with the shock at the exit plane to estimate the shock's location:

$$L_{shock} \approx 1.75 \text{ m} + \left(\frac{350 - 367.7}{330.8 - 367.7}\right)(3.5 - 1.75) \approx 2.59 \text{ m}$$

We can use this value to re-compute the flow and find the exit Mach number, or we can linearly interpolate our solutions from (b) and (d) to estimate the exit Mach:

$$M_{exit} \approx 0.705 + \left(\frac{350 - 367.7}{330.8 - 367.7}\right)(0.776 - 0.705) \approx 0.739$$

(e) Direct Solution for Exit Mach Number

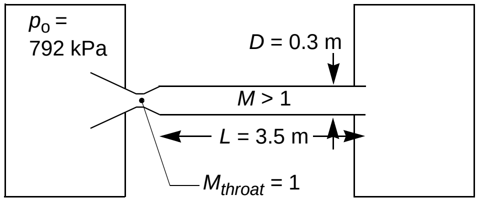
Note that, once we have identified that the shock is located in the channel (steps (b) and (c) above), it is not necessary to iterate to find the conditions at the channel exit. Since we know that the exit flow is subsonic in this case, $p_{exit} = p_{back} = 350$ kPa. Further, since we know from the conditions at the entrance of the channel, $p^* = 248.3$ kPa, and this reference state applies throughout. Thus, the exit Mach number must be:

$$\frac{p}{p^*} = \frac{350}{248.3} = 1.4095 \Rightarrow M_{exit} = 0.738$$

Thus, our estimate from linear interpolation in (d) was very close to the exact answer. It is still necessary to iterate in order to find the location of the shock.

(f) Shock at Throat (Subsonic Flow in Diverging Nozzle)

Although not explicitly asked for in this problem, it is of interest to also examine the case where the reservoir pressure remains fixed and the back pressure has been increased so that the normal shock has been forced to the throat of the converging-diverging nozzle and that subsonic flow exists throughout.




Since the channel to throat area ratio remains the same and the flow is sonic at the throat but subsonic in the diverging section:

$$\left(\frac{A}{A^*}\right) = 1.6875 \Rightarrow M_{inlet} = 0.372$$

$$\left(\frac{p}{p_o}\right) = 0.9089 \Rightarrow p_{inlet} = 720.4 \text{ kPa}$$

We can now solve the problem of a subsonic frictional flow entering a channel 3.5 m long at Mach 0.372:

$$\left(\frac{4fL_1^*}{D_H}\right)_{M\,=\,0.372} = 2.8867\,,\ \left(\frac{p}{p^*}\right)_{M\,=\,0.372} = 2.9051 \Rightarrow p^* = 248.0 \text{ kPa}$$

Note that $p^*$ is still 248 kPa (the same value if supersonic flow exists in the diverging nozzle). This is because, as long as the stagnation pressure is fixed and the mass flow rate remains constant, the sonic condition for frictional flow will be constant.

$$\frac{4fL_2^*}{D_H} = \left(\frac{4fL_1^*}{D_H}\right)_{M_y\,=\,0.655} - \frac{4fL_{1-2}}{D_H} = 2.8867 - \frac{4 \cdot 0.005 \cdot 3.5}{0.3} = 2.6534$$

$$\frac{4fL_2^*}{D_H} = 2.6534 \Rightarrow M_2 = 0.383,\ \frac{p}{p^*} = 2.823 \Rightarrow p_2 = 700.1 \text{ kPa}$$

Thus, the exit pressure in this case would be $p_{exit} = 700.1$ kPa.

**Summary:** We can summarize our results with variable back pressure below:

- $p_o > p_{back} > 700$ kPa $\Rightarrow$ flow subsonic everywhere ($\dot{m}$ depends on $p_{back}$)
- 700 kPa $> p_{back} > 408$ kPa $\Rightarrow$ flow sonic at throat; normal shock in diverging nozzle (now choked: $\dot{m}$ independent of $p_{back}$)
- 408 kPa $> p_{back} > 331$ kPa $\Rightarrow$ supersonic flow at channel entrance; normal shock in channel
- 331 kPa $> p_{back} > 177$ kPa $\Rightarrow$ supersonic flow through entire channel; oblique shock waves outside channel
- 177 kPa $> p_{back}$ $\Rightarrow$ supersonic flow through entire channel; expansions outside channel.

Thus, for a given back pressure, we can immediately identify the existence and location of the normal shock. For example, we if are told the back pressure is 500 kPa, we immediately know there is a normal shock in the diverging section of the nozzle. It is then simply a matter of iterating on its location so that the exit pressure $p_{exit} = p_{back} = 500$ kPa.