Compressible Fluids Notes

Lecture Notes for MECH-430: Fluids II

by

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1. Introduction

Compressible Fluid Dynamics refers to fluid flow in which the velocity of the fluid motion is sufficient to "squeeze" and "stretch" the fluid, altering its density. Thus, we can no longer use the incompressible assumption of constant density that was so essential in developing the relations of low velocity fluid dynamics (e.g. Bernoulli's equation). Now, each element of the fluid becomes like a tiny "piston & cylinder" problem from thermodynamics, in which work is done on the fluid element by the neighboring fluid elements. Because the fluid element is being compressed and expanded, we must introduce the First Law of Thermodynamics (i.e., the Energy Equation) in our development of the analysis. In compressible fluid flow, the Momentum Equation is now coupled to the Energy Equation, so we have to solve them together. In this sense, compressible fluid dynamics is the marriage between fluid dynamics and thermodynamics. This coupling complicates the analysis, but results in a methodology of solving fluid dynamics problems that is more general than incompressible fluid dynamics, namely, we can now treat problems with high speed flow where the incompressible assumption breaks down.

Note that the use of the term "fluid" can refer to either a liquid or a gas. As we know from thermodynamics, the distinction between liquid and gas can become blurry, especially when we are in the supercritical region of the phase diagram. The mathematical formulation of the basic conservation laws (conservation of mass, momentum, and energy) are identical for liquids and gases; they only differ in the thermodynamic equation of state. So, we use the term "fluid" to describe both liquids and gases, but in these notes, the emphasis will definitely be on gases (gas dynamics and aerodynamics).

1.1 Applications

Compressible fluid dynamics has applications wherever high-speed fluid motion is encountered. What we mean by "high-speed" (i.e., high relative to what?) will be addressed in greater detail in Chapters 4-5, but suffice for now that when the fluid velocity becomes comparable to the sound speed of the fluid, we must treat the fluid as compressible. Obvious applications can be found in aerospace engineering, in both external and internal flows. External flows refer to the aerodynamic flow over a wing or body. Internal flows refer to flows that pass through a device, such as jet engine or a rocket. So, our list of applications begins with:

- high-speed aerodynamics of wings and bodies
- jet propulsion, gas turbines
- rocket engines
- flow through valves, inlets, and exhausts found in internal combustion engines

Other applications include:

- gas flows in pipelines

and applications of combustion and chemically reacting flows:

- high speed flames
- detonations and explosions
- gas flows in lasers

Compressible fluid dynamics is also encountered by physicists who study high-speed flows in astronomical contexts:

- astrophysics (solar wind from the sun, supernovae, plasma jets from stars, extragalactic jets from active galaxies and quasars, etc.)

One of the most unique features of compressible flow, *shock waves*, show up in a surprising number of engineering applications and natural phenomenon, such as sonic booms, volcanic eruptions, asteroid impacts, etc. Shock waves are used as "tools" by engineers, chemists, physicists, and geologists in the laboratory to create extreme states of pressure, temperature, or velocity. Shock waves are even being used in medical procedures, such as "Extracorporeal Shock Wave Lithotripsy," which is used as an alternative to surgery to destroy kidney stones by focusing shock waves through the body. In aerospace applications (supersonic inlets and nozzles), we usually try to avoid the formation of shock waves, as they represent a loss in efficiency. In the operation of real-world devices, however, shock wave formation is impossible to avoid.

Finally, there are other fields that are technically not compressible fluid flow but that can be shown to be mathematically equivalent to the dynamics of compressible fluids. These fields of research are now benefiting from "transferring" the method of analysis developed for compressible fluid flow:

- shallow water waves
- traffic flow and traffic jams

We will use these concepts to illustrate various features of compressible fluid flow in this course.

1.2 Treatment

These notes will develop a treatment of the topics discussed above starting with only very fundamental concepts from mechanics ($F = ma$) and thermodynamics (First Law). The emphasis is on illustrating how fundamental conservation laws (mass, momentum, energy) dictate all of compressible fluid behavior. The initial challenge will be to formulate these tools in a way so that they can be applied to a fluid that is flowing through an open system. Once we have our tools in control volume form for open systems, we can proceed quickly to examine a number of interesting applications.

The development followed here is "fast and loose," designed to give a feel for compressible fluids as quickly as possible, and then to move on to applications that illustrate the underlying physics. This treatment is very different than the usual textbook progression, which exhaustively develops the theory first and only examines applications as an afterword (if at all). In my teaching of a Compressible Fluids course at McGill University, I have found student interest and enthusiasm are better maintained if applications are integrated as early as possible into the material. This approach means that some examples, such as rocket nozzles, are introduced before we have the complete tool set (e.g., normal and oblique shock relations) required to completely analyze nozzle flow. Thus, certain problems are periodically revisited throughout the notes as our ability to treat different flow scenarios increases. While this need to return to the nozzle example throughout the notes may appear cumbersome, I believe this approach properly reflects the "converging-diverging nozzle with variable back pressure" example as the canonical problem in compressible flow.

These notes also introduce some topics, such as Mach reflection, that are not usually included in undergraduate compressible fluid flow courses. The purpose here is to give the student some hints as to were the limits of our current understanding of compressible flow might be encountered, and to provide some hope for the aspiring researcher that not everything in compressible flow has already been done.

1.3 Further Reading

For a more rigorous, formal treatment, the following text books are highly recommended:

- Saad, M., *Compressible Fluid Flow*, 2nd edition, Prentice-Hall, Inc., 1993.
- Oosthuizen, P. H. and Carscallen W.E., *Compressible Fluid Flow*, McGraw-Hill, 1997.
- Anderson, J.D., *Modern Compressible Flow with Historical Perspective*, 3rd Edition, McGraw-Hill, 2004.
- Zucker, R.D., and Biblarz, O., *Fundamentals of Gas Dynamics*, 2nd Edition, John Wiley & Sons, Inc., 2002.

The Anderson book is particularly recommended for the interesting historical narratives found throughout the text. More thorough treatments, considering a wide range of flows, are found in:

- Zucrow, M.J., Hoffman, J.D., *Gas Dynamics*, Vols. 1 and 2, Wiley, 1976.
- Owczarek, J.A., *Fundamentals of Gas Dynamics*, International Textbook Company, 1968.

The Zucrow and Hoffman book is a useful resource for expanding the treatment found in these notes to include the effects of imperfect gas effects (i.e., variable specific heats) and chemical reaction. The Owczarek book (long out of print) is broader in scope than most compressible flow texts, detailing a variety of flows not found in other books.

Students expecting to work with compressible flows should also have at least a cursory familiarity with the "classic" texts of the field:

- Shapiro, A., *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Vols. 1 and 2, Wiley, 1953.
- Liepmann, H. W., and Roshko, A., *Elements of Gas Dynamics*, Dover, 2002.

The Shapiro's book (and name) is synonymous with internal compressible flow for several generations of researchers. The book by Liepmann and Roshko (originally published in 1957) focuses more on external flows and is now back in print in an affordable Dover edition.

1.4 Acknowledgements

I am profoundly indebted to the individuals who contributed to my own understanding of compressible fluid flow. Each of these individuals' contributions (via lectures, discussions, or their own course notes) is reflected in these notes: Charles Bond, Harold Barthel, John Buckmaster, Shee Mang Yen (University of Illinois, Urbana/Champaign), Abraham Hertzberg, Scott Eberhardt, Carl Knowlen, Adam Bruckner (University of Washington, Seattle), John H.S. Lee and Rom Knystautas (McGill University, Montreal).

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Despite the assistance of many people, the errors that remain are my own.

Andrew Higgins
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2. Developing the Equations of Fluid Flow

We enter the study of compressible fluid dynamics with a number of tools already in our tool box.

Perhaps most fundamental is the conservation of mass:

$$m_{closed\ system} = \text{constant} \tag{2.1}$$

This states that mass is neither created or destroyed, so if we limit our attention to a *closed system* (i.e., a system in which mass neither enters nor leaves), then the mass must remain fixed. The conservation of mass is also known as "continuity."

Another tool we have is the basis of most engineering courses: Newton's Second Law:

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \tag{2.2}$$

Recall that this is vector equation, so it is really three equations in one: one equation for each direction (x, y, z). Another way Newton's Second Law can be written is:

$$\Sigma\vec{\boldsymbol{F}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{\boldsymbol{v}}) \tag{2.3}$$

In words, this states, "The rate of change of momentum is equal to the net force applied." This means the momentum is conserved, unless a force is acting. Thus, this is also a conservation law, the "conservation of momentum."

From basic thermodynamics, we have the First Law:

$$\Delta E = \delta W + \delta Q \tag{2.4}$$

which states that: "The change of energy of a closed system equals the heat transfer to the system plus the work done on the system."* The First Law of Thermodynamics can also be referred to as "conservation of energy."

In addition to the three basic conservation laws, we also have some other concepts from thermodynamics, such as the equation of state that relates the various thermodynamic parameters like temperature, pressure, density, internal energy, etc. For compressible fluid dynamics, we can usually apply the ideal gas equation of state:

$$p = \rho R T \tag{2.5}$$

In general, an equation of state is a function of the form:

$$p = p(\rho, T)$$

$$u = u(\rho, T)$$

$$h = h(\rho, T)\text{, etc.}$$

Note that specifying the value of two thermodynamic variables (ρ and T) determines the rest. Alternatively, we could take p and T as the independent variables. If the ideal gas equation of state does not apply, then determining the exact form of these functional dependences can be very challenging. For this course, we will limit ourselves to the ideal gas equation of state. Further, we will consider calorically perfect gases, for which c_p and c_v are constant. Thus, $h = c_pT$ and $u = c_vT$.

Also, from thermodynamics, we learned of a property called entropy, and that for an isolated system, entropy can never decrease. In certain idealized cases, the entropy can be constant, and we refer to processes that maintain constant entropy as isentropic:

*Note that the sign of δQ and δW may be different, depending on which thermodynamics textbook you used. Here, we will take δQ and δW as being heat transfer *to* the system and work done *on* the system, respectively.

$$ds_{isentropic} = 0 \tag{2.6}$$

These basic concepts form the "toolbox" from which we will build an analysis of compressible fluid flow. If you are not completely comfortable with any of these concepts, now is the time to review them from your thermodynamics textbook before proceeding further.

Toolbox	
mass:	$m_{closed\ system} = \text{constant}$
momentum:	$\Sigma\vec{F} = \dfrac{d}{dt}(m\vec{V})$
energy:	$\Delta E = \delta W + \delta Q$
ideal gas:	$p = \rho RT$
isentropic:	$ds_{isentropic} = 0$

We would like to apply these concepts to the flow of a continuous fluid (air, water, etc.). The picture we should have in mind is a stream of water or a jet of air. It is not clear how we can directly apply the equations in our tool box above to this situation, since it is difficult to define a closed system of fixed mass in this case.

Consider for the moment a room with open windows and doors, but wire screens in the windows and doorways. This room has a well defined volume, but since air can flow in and out the windows and doors, the mass is not constant. If a cold weather front were to pass through, cold, dense air may blow into the room and increase the total mass of air. If a warm weather front passes through, warm, low density air may came in, lowering the total mass of air. If we want to apply the conservation of mass, we cannot do so directly. We need a version for a control volume.

This approach is called the "Eulerian" approach, and it is particularly suited to compressible flow applications like jet engines or rockets. Typically we are interested in fluid properties at a point (for example, the leading edge of a wing) or the properties over a volume (e.g., the rate of mass flow though an engine). The alternative approach is called the "Lagrangian" approach, where we track specific, individual elements of the fluid. But usually we do not care what happens to a fluid element after it has passed over our wing or through our engine, and the Lagrangian approach requires excessive bookkeeping to keep track of all the various elements of the flow.*

2.1 Conservation of Mass for a Control Volume

So, we need a version of the conservation of mass for an open system (control volume, defined by the dashed line drawn below).

Initially, consider an instant in time t, when the mass in the control volume is $m_{cv}(t)$, and the mass that is about to flow into the control volume is Δm_{in}. At some time Δt later, Δm_{in} as gone into the C.V., and a mass Δm_{out} has come out.

Note that we chose a Δt such to correspond to the exact amount of time required for Δm_{in} to be swept into the C.V. Now, if we define a *fixed mass system* consisting of the mass initially in the C.V. at time t and the mass that will be swept in during time Δt, then this is the same fixed mass in the C.V. at time $t + \Delta t$ and the mass that has left. The mass that comprises this fixed mass system is shaded grey here. In an actual experiment, we might put dye in water or radioactive tracer particles in air to help track this system of fixed mass. Since mass is conserved:

$$m_{cv}(t) + \Delta m_{in} = m_{cv}(t+\Delta t) + \Delta m_{out}$$

Rearranging:

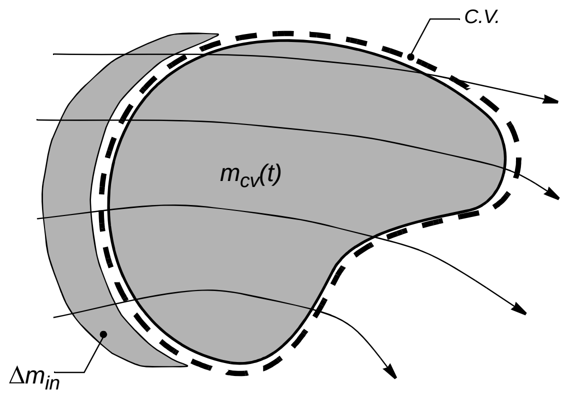
$$m_{cv}(t+\Delta t) - m_{cv}(t) = \Delta m_{in} - \Delta m_{out}$$

and dividing by dt:

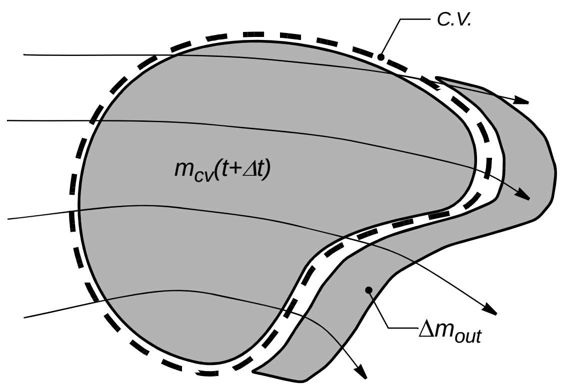
$$\frac{m_{cv}(t+\Delta t) - m_{cv}(t)}{\Delta t} = \frac{\Delta m_{in}}{\Delta t} - \frac{\Delta m_{out}}{\Delta t}$$

*There are instances where the Lagrangian approach is desirable, for example, if we are studying the motion of a single droplet of fuel injected into an engine, the Lagrangian formulation may make tracking the fuel droplet easier.

at time t



at time $t + \Delta t$



in the limit as $\Delta t \to 0$, the first term becomes the derivative (time rate of change) of the mass in the control volume, and the $\frac{\Delta m_{in}}{\Delta t}$ and $\frac{\Delta m_{out}}{\Delta t}$ terms represent the rate [kg/s] at which mass is entering and leaving the control volume. Thus,

$$\frac{\partial m_{cv}}{\partial t} = \dot{m}_{in} - \dot{m}_{out}$$

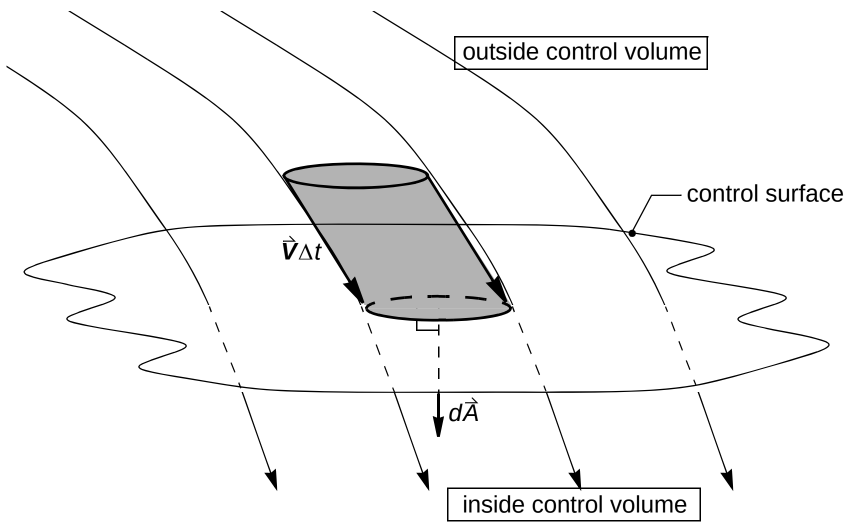
In general, there may be many sources of mass entering or leaving (think of a building with many windows and doors):

$$\frac{\partial m_{cv}}{\partial t} = \sum \dot{m}_{in} - \sum \dot{m}_{out}$$

In words, this states: "The rate of change of mass in a control volume equals the total rate of mass flow in minus the total rate of mass flow out." This makes sense because, by the conservation of mass, all the mass must go somewhere. If the total rate going in does not equal the total rate of mass flow out, then mass must either be accumulating or depleting in the control volume.

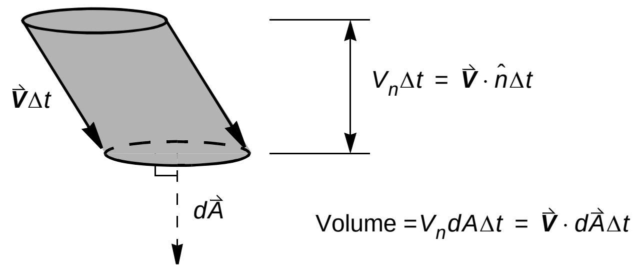
We would like to express the conservation of mass in terms of *flow variables* like density, velocity, etc. The mass in the control volume is just the integral of density over the control volume: $m_{cv} = \int_{CV} \rho dV$

The rate of mass flow across the control surface that defines the control volume can be determined by examining an element of surface area $d\vec{A}$.



Note that $d\vec{A}$ is a vector, whose direction always points normal to the surface (unit vector $\hat{n}$) into the control volume by definition. The magnitude of $d\vec{A}$ is just the area of that surface element [m^2]. The amount of mass flow through that element in a time Δt is given by the mass of fluid in a cylinder of length $\vec{\mathbf{V}}\Delta t$:

Recall the volume of a cylinder is its base area times its height (not the length of the side of the cylinder):



We can also express this volume using vector notation $\vec{\mathbf{V}} \cdot d\vec{A}\Delta t$. The mass in the cylinder is just the density times the volume (here, we will assume the cylinder is so small, it can be represented by a single value of density and velocity): $dm = \rho\vec{\mathbf{V}} \cdot d\vec{A}\Delta t$ Dividing by Δt, the rate at which mass flows across the surface element dA is $\rho\vec{\mathbf{V}} \cdot d\vec{A}$.

Note: What if mass is flowing *out* of the control surface? In this case, $\vec{\mathbf{V}} \cdot d\vec{A}$ < 0 by the rule of vector multiplication, so that $\dot{m}_{in} < 0$. A "negative inflow of mass" is the same as a "positive outflow", so the $\vec{\mathbf{V}} \cdot d\vec{A}$ expression automatically accounts for mass flow leaving the C.V. This applies to just one infinitesimally small element of the surface. We can sum this over the entire control surface by a surface integral

$$\sum \dot{m}_{in} - \sum \dot{m}_{out} = \int_{CS} \rho\vec{\mathbf{V}} \cdot d\vec{A}$$

Thus, the conservation of mass becomes:

$$\boxed{\frac{\partial}{\partial t}\int_{CV} \rho dV = \int_{CS} \rho\vec{\mathbf{V}} \cdot d\vec{A}}$$

And we can replace

$$m_{closed\ system} = \text{constant}$$

with this new, open-system control-volume version in our toolbox.

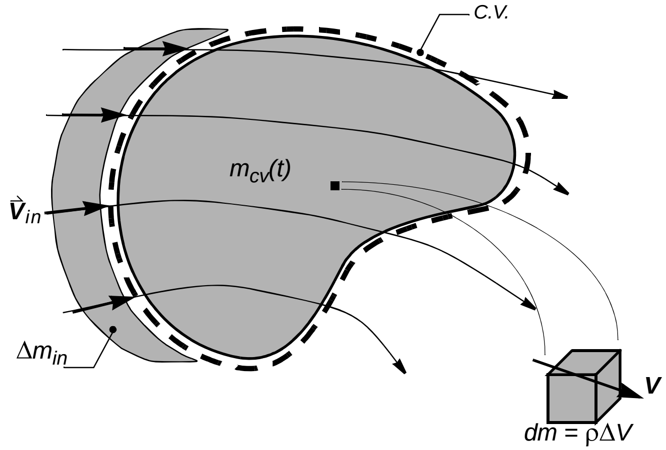
2.2 Conservation of Momentum for a Control Volume

Taking a similar approach for momentum, we will consider the mass entering and leaving the control volume between time t and $t + \Delta t$.

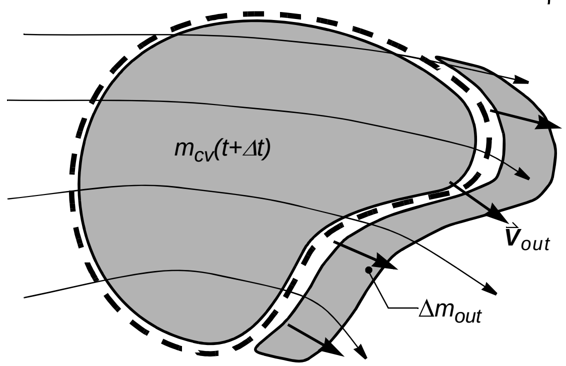
Note that the control volume is comprised of infinitesimal elements of volume dV, each of which has momentum $\rho\vec{\mathbf{V}}dV$. Note that $\vec{\mathbf{V}}$ is the velocity vector, while V is volume. Therefore, the total momentum of the fixed control mass at time t is $\int_V \rho(t)\vec{\mathbf{V}}(t)dV + \Delta m_{in}\vec{\mathbf{V}}_{in}$, and the momentum at time $t + \Delta t$ is $\int_V \rho(t+\Delta t)\vec{\mathbf{V}}(t+\Delta t)dV + \Delta m_{out}\vec{\mathbf{V}}_{out}$. Thus, the difference between these two:

$$\Delta(\text{total momentum}) = \int_V \rho(t+\Delta t)\vec{\mathbf{V}}(t+\Delta t)dV - \int_V \rho(t)\vec{\mathbf{V}}(t)dV - (\Delta m_{in}\vec{\mathbf{V}}_{in} - \Delta m_{out}\vec{\mathbf{V}}_{out})$$

at time t

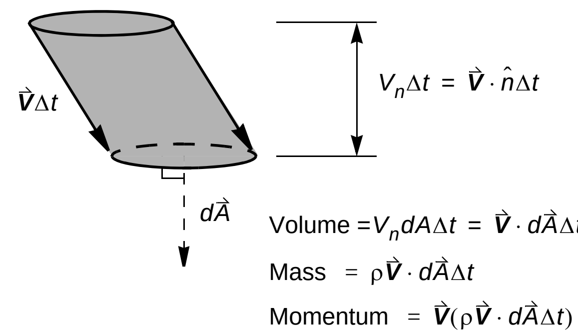


at time $t + \Delta t$



Dividing both sides by Δt and taking $\Delta t \to 0$, we use Newton's second law $\frac{\mathrm{d}}{\mathrm{d}t}(\text{total momentum}) = \Sigma\vec{F}_{CV}$, where $\Sigma\vec{F}_{CV}$ is the force acting on the control volume.*

In general, there may be multiple mass inlets and outlets with different velocities, so we will again use an integral over surface elements



The net flux of momentum into the control volume is

$$\sum \dot{m}_{in}\vec{V}_{in} - \sum \dot{m}_{out}\vec{V}_{out} = \int_{CS} \vec{V}(\rho\vec{V}\cdot d\vec{A})$$

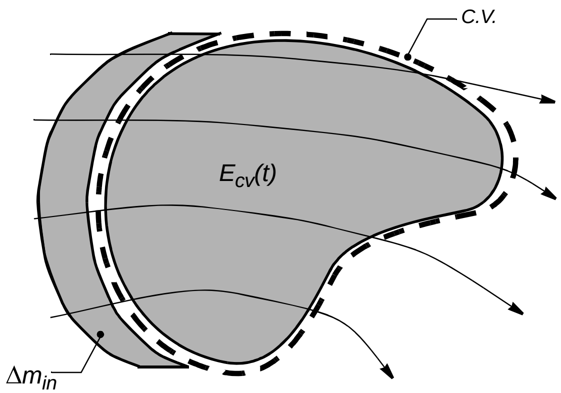
Note that this expression is a vector, since as mass is convected into and out of the control volume, it carries momentum with x, y, and z components. Substituting this expression into the equivalence of force and the rate of change of momentum, we get

$$\boxed{\Sigma\vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho\vec{V}dV - \int_{CS} \vec{V}(\rho\vec{V}\cdot d\vec{A})}$$

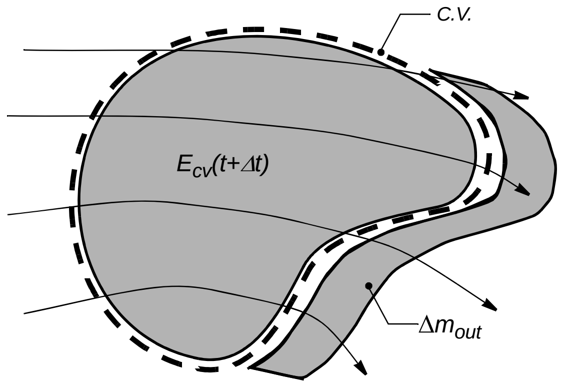
In words, this equation states that: "The sum of forces acting on a control volume equals the rate of change of momentum inside the control volume minus the rate at which momentum is being convected into the control volume." This will replace $\Sigma\vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{V})$.

*Note that technically, we should be using the net force acting on the *control mass* (consisting of mass in control volume and mass entering/leaving control volume), but since we are examining the limit as $\Delta t \to 0$, the discrepancy will vanish in the limit.

at time t



at time $t + \Delta t$



2.3 Conservation of Energy for a Control Volume

Finally, consider the energy equation (First Law of Thermodynamics) and a flow passing though a control volume.

The energy in the control volume at time t is $E_{cv}(t)$. Here, E represents the total energy (internal, kinetic, etc.). The mass that is about to be carried into the C.V. has mass Δm_{in}, specific internal energy u_{in}, and velocity $\vec{V}_{in}$. Thus the total energy of the closed mass system (mass shaded in figure) is:

$$E(t) = E_{cv}(t) + \Delta m_{in}\left(u_{in} + \frac{1}{2}V_{in}^2 + \ldots\right)$$

Note that, in general, the energy term of the mass entering the control volume can also have gravitational potential energy ($\Delta m_{in} gh$) and other forms of energy, but in most applications of compressible flow, potential energy can be neglected.

At time $t + \Delta t$, the total energy is:

$$E(t+\Delta t) = E_{cv}(t+\Delta t) + \Delta m_{out}\left(u_{out} + \frac{1}{2}V_{out}^2 + \ldots\right)$$

First Law of Thermodynamics becomes:

$$E(t+\Delta t) - E(t) = E_{cv}(t+\Delta t) - E_{cv}(t) + \Delta m_{out}\left(u_{out} + \frac{1}{2}V_{out}^2\right) - \Delta m_{in}\left(u_{in} + \frac{1}{2}V_{in}^2\right)$$

$$= \delta W + \delta Q$$

dividing by Δt and rearranging:

$$\frac{E_{cv}(t+\Delta t) - E_{cv}(t)}{\Delta t} = \frac{\delta W}{\Delta t} + \frac{\delta Q}{\Delta t} + \frac{\Delta m_{in}}{\Delta t}\left(u_{in} + \frac{1}{2}V_{in}^2\right) - \frac{\Delta m_{out}}{\Delta t}\left(u_{out} + \frac{1}{2}V_{out}^2\right)$$

In the limit as $\Delta t \to 0$:

$$\frac{\partial E_{cv}}{\partial t} = \dot{W} + \dot{Q} + \dot{m}_{in}\left(u_{in} + \frac{1}{2}V_{in}^2\right) - \dot{m}_{out}\left(u_{out} + \frac{1}{2}V_{out}^2\right)$$

Where $\dot{Q}$ and $\dot{W}$ are the rate of heat transfer and work to the system.

Considerable caution must be exercised in treating the term $\dot{W}$. Typically, we think of work as being done by deforming the boundaries of a system (PdV work) or work being done by a shaft sticking into a system (or other interactions, such as electromagnetic fields, etc.). Here, the boundaries of the C.V. are fixed, but the First Law as written here applies to the *fixed mass system* moving though the C.V., and there is *PdV* work associated with the motion of the m_{in} and m_{out} masses. Thus, there ***is*** work being done in pushing mass in and out of the C.V., even if the C.V. boundaries do not move.

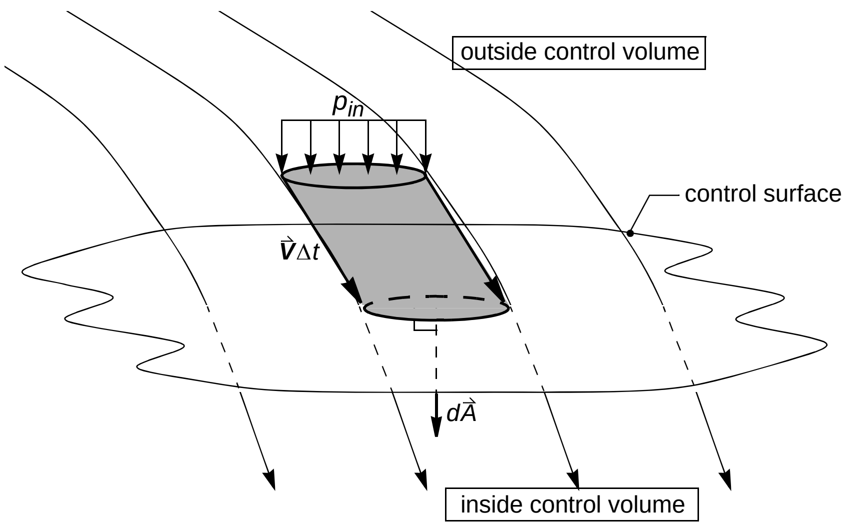
For this reason, it becomes convenient to split the $\dot{W}$ work into two terms:

$$\dot{W} = \dot{W}_{cv} + \dot{W}_{\text{flow work}}$$

$\dot{W}_{cv}$ refers to work done on the control volume by shafts, electric fields, etc., and $\dot{W}_{\text{flow work}}$ refers to work done in pushing mass into and out of the C.V.

What is the work done on the fixed mass system as the mass *dm* enters the C.V.? Work is given by *pdV*

$$dW_{\text{flow work}} = pdV = p_{in}\vec{\boldsymbol{V}} \cdot d\vec{A}\Delta t$$



We can divide by Δt to get the rate at which flow work is done. Note that the volume term $\vec{\boldsymbol{V}} \cdot d\vec{A}$ can be expressed in terms of the mass inflow as $\frac{\dot{m}_{in}}{\rho_{in}}$. Thus, the flow work term becomes:

$$\frac{dW_{\text{flow work}}}{dt} = p_{in}\vec{\boldsymbol{V}} \cdot d\vec{A} = p_{in}\frac{\dot{m}_{in}}{\rho_{in}}$$

Substituting this expression for the flow work into the First Law of an open control volume:

$$\frac{\mathrm{d}E_{cv}}{\mathrm{d}t} = \dot{W}_{cv} + \dot{Q} + \dot{m}_{in}\left(u_{in} + \frac{p_{in}}{\rho_{in}} + \frac{1}{2}\vec{\boldsymbol{V}}_{in}^2\right) - \dot{m}_{out}\left(u_{out} + \frac{p_{out}}{\rho_{out}} + \frac{1}{2}\vec{\boldsymbol{V}}_{out}^2\right)$$

Note the appearance of the term $u + \frac{p}{\rho} = u + pv$, which is called ***enthalpy***. Enthalpy, denoted *h*, is introduced in thermodynamics because it takes into account the work associated with bringing energy into a control volume. Since flow work is now taken care of by the use of the enthalpy term, the only explicit work term is $\dot{W}_{cv}$, which includes only the work input into the control volume via shaft work, etc.

$$\frac{\mathrm{d}E_{cv}}{\mathrm{d}t} = \dot{W}_{cv} + \dot{Q} + \dot{m}_{in}\left(h_{in} + \frac{1}{2}\vec{\boldsymbol{V}}_{in}^2\right) - \dot{m}_{out}\left(h_{out} + \frac{1}{2}\vec{\boldsymbol{V}}_{out}^2\right)$$

Again, we will express this using fluid dynamic variables as an integral over the control volume (for E_{cv}) and over the control surface (for $\dot{m}_{in}$ and $\dot{m}_{out}$).

$$\boxed{\frac{\partial}{\partial t}\int_{CV} \rho e dV = \dot{W}_{cv} + \dot{Q} + \int_{CS}\left(h + \frac{\vec{\boldsymbol{V}}^2}{2}\right)\rho\vec{\boldsymbol{V}} \cdot d\vec{A}}$$

e is the mass-specific total energy. In words, this equation states: "The rate of change of the total energy in a control volume equals the rate of work done on the control volume *plus* the rate of heat transfer to the control volume *plus* the rate at which mass flow carries enthalpy and kinetic energy into the control volume."

Our toolbox now appears as

Toolbox

mass: $$\frac{\partial}{\partial t}\int_{CV} \rho dV = \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

momentum: $$\Sigma \vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{V} dV - \int_{CS} \vec{V}(\rho \vec{V} \cdot d\vec{A})$$

energy: $$\frac{\partial}{\partial t}\int_{CV} e\rho dV = \dot{W}_{cv} + \dot{Q} + \int_{CS} \left(h + \frac{\vec{V}^2}{2}\right)\rho \vec{V} \cdot d\vec{A}$$

ideal gas: $p = \rho R T$

isentropic: $ds_{isentropic} = 0$

Collectively, the conservation laws of mass, momentum, and energy are called "Euler Equations," although technically only the momentum equation comes from Euler's equation $\Sigma \vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{V})$. Euler himself derived the mass and momentum equations in 1753; the energy equation had to await the development of thermodynamics in the 19th century.

If we include the effects of viscosity and heat transfer via molecular transport in deriving these conservation equations, we obtain the more general Navier-Stokes equations. These equations can be said to contain all of fluid dynamics. Fortunately for the job security of fluid dynamicists, they are not so easy to solve. They are, after all, a coupled set of nonlinear partial differential equations!

Leonhard Euler (1707-1783)

For this course, we will consider mostly inviscid fluid dynamics, so the Euler equations apply. These in general are also difficult to solve, but computational fluid dynamics (CFD) is increasing its capabilities to solve the Euler equations quickly and with confidence that the solution generated numerically is the correct one. However, while numerical solutions can solve a specific problem, they cannot give us a conceptual understanding of the effects and trends that dominate compressible fluid flow. We need to be able to solve the Euler equations (or at least, a simplified version of them) analytically if we are to obtain a "feel" for how compressible flow responds. So, we will next turn to one-dimensional flows that permit the equations to be solved exactly.

Leonhard Euler appears on the Swiss 10 Franc note.

Even when we do solve the full, three-dimensional Euler equations numerically, the sheer volume of numerical data generated often forces us to average the results back into a one-dimensional representation of the flow, just so that we can visualize and understand the results. Thus, one-dimensional flow will always comprise a large amount of the treatment of compressible flow.

Most pragmatically, the results of our one-dimensional analysis will be a set of tools that can be applied to solve a number of complex engineering problems with surprising accuracy.

3. Equations of One-Dimensional, Steady Flow

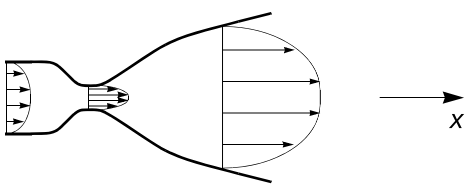
The conservation laws we have developed:

mass: $$\frac{\partial}{\partial t}\int_{CV} \rho\, dV = \int_{CS} \rho \vec{\boldsymbol{V}} \cdot d\vec{A}$$

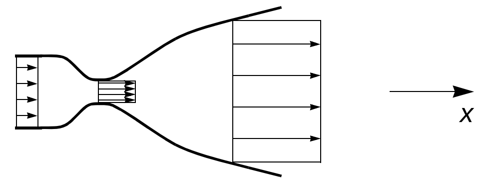
momentum: $$\Sigma \vec{\boldsymbol{F}} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{\boldsymbol{V}} dV - \int_{CS} \vec{\boldsymbol{V}}(\rho \vec{\boldsymbol{V}} \cdot d\vec{A})$$

energy: $$\frac{\partial}{\partial t}\int_{CV} \rho e\, dV = \dot{W}_{cv} + \dot{Q} + \int_{CS}\left(h + \frac{\vec{\boldsymbol{V}}^2}{2}\right)\rho \vec{\boldsymbol{V}} \cdot d\vec{A}$$

...contain all of compressible fluid dynamics. But they cannot be solved analytically. If we restrict our attention to one-dimensional flows, however, analytic solutions are possible. What do we mean by one-dimensional? Consider flow through a rocket nozzle:

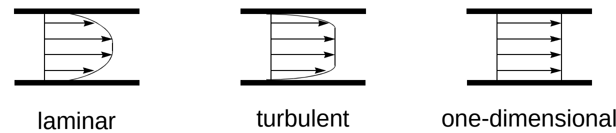


We know from our earlier studies of fluids that all fluids have some viscosity, so the velocity of the flow near the wall is always zero relative to the wall. This results in a velocity distribution across the nozzle as shown here. We can idealize the flow, however, so that the velocity profiles are only dependent on one variable (here, the *x*-coordinate) and the flow properties are constant across the cross section.



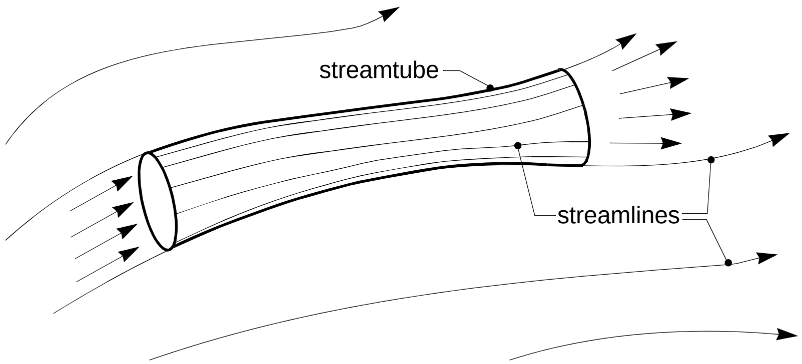
This approximation is valid for:

- Slow variations in area change
- Turbulent flow more than laminar



Of course, no real flow is one-dimensional; it is an idealization we make to facilitate solving the governing conservation equations. Remarkably, however, a number of internal flows (rockets, jet engines, etc.) can be modeled as one-dimensional with a good degree of accuracy. Some external flows can also be treated as one-dimensional if we examine *stream tubes*. A stream tube is a control volume defined by stream lines; it is an imaginary tube whose walls are always parallel to the flow. A *stream line*, in turn, is a line that is always tangent to the local velocity vector. In a steady flow, particles travel along stream lines. We have already been drawing streamlines in Chapter 2.

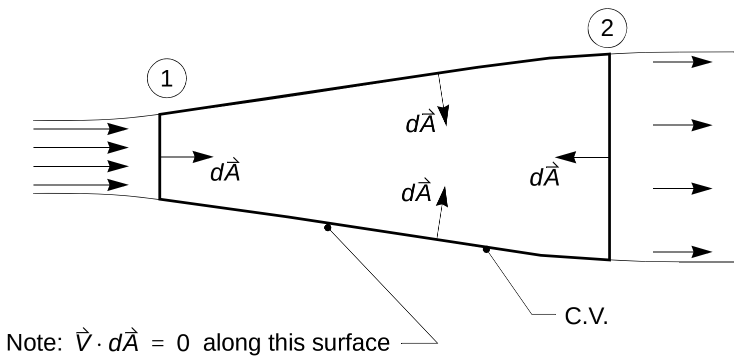
A streamtube of infinitesimally small cross-sectional area can be treated as a one-dimensional flow.



3.1 Conservation of Mass for 1-D Stream Tube

Starting with continuity: $\dfrac{d}{dt}\int_{CV}\rho dV = \int_{CS}\rho\vec{V}\cdot d\vec{A}$

We can specialize this equation for a one-dimensional stream tube:



Since the flow is parallel to the walls of the stream tube, there is no mass flux through the walls. The only mass flow is through the ends of the streamtube (denoted "1" and "2" here). Thus, continuity becomes:

$$\frac{d}{dt}\int_{CV}\rho dV = \rho_1 \boldsymbol{V}_1 A_1 - \rho_2 \boldsymbol{V}_2 A_2$$

If the flow is steady, then the $\dfrac{d}{dt}\int_{CV}\rho dV = 0$, since nothing is varying in time.

Note the terminology we use:

- What do we mean by *steady*? Steady means "constant in time" or "not varying in time."
- What do we mean by *uniform*? Uniform means "constant in space" or "not varying in space." The flows we are studying here are uniform in cross section across the stream tube.

Thus, it is good to refrain from saying that "a flow is constant" because this is not a very precise statement, rather we say it is steady (if constant in time) or uniform (if constant in a spatial dimension).

Our steady, one-dimensional continuity equation becomes

steady, 1-D

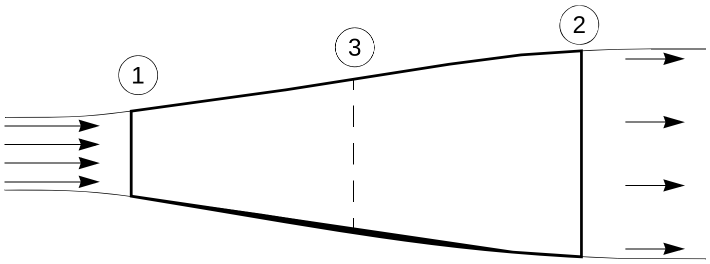
$$\boxed{\rho_1 \boldsymbol{V}_1 A_1 = \rho_2 \boldsymbol{V}_2 A_2}$$

Note that this equation is valid even if the flow between stations 1 and 2 is not one-dimensional.

If the flow is one-dimensional over the entire control volume, then this equation applies at every point within the control volume, for example, at a station 3 between 1 and 2. So:

$$\rho_1 \boldsymbol{V}_1 A_1 = \rho_2 \boldsymbol{V}_2 A_2 = \rho_3 \boldsymbol{V}_3 A_3$$

Since station 3 is at an arbitrary location, ρVA must be constant over the entire control volume. Therefore, another expression of continuity is:



steady, 1-D

$$\rho \boldsymbol{V} A = \text{constant}$$

This is an integral, or control volume, form of the conservation of mass. It is very useful for problem solving for flows passing though a fixed volume, but in order to obtain a qualitative feel for how flow responds to changes, it is helpful to examine a differential version of the conservation law that applies at a point or to an infinitesimal section of the control volume. We will also require a differential version of the conservation law for cases in which we cannot solve the flow across a control volume, but must instead integrate the governing equations through the flow.

To obtain a differential version of the conservation law from the integral from, we can apply the differential operator:

$$d(\rho \boldsymbol{V} A) = d(\text{constant})$$

$$VAd\rho + \rho AdV + \rho VdA = 0$$

Dividing by ρVA:

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

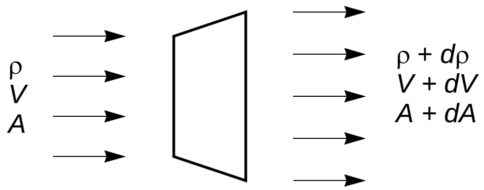
We can obtain this result more directly via logarithmic differentiation:

$$\ln(\rho \boldsymbol{V} A) = \ln(\text{constant})$$

$$d(\ln(\rho \boldsymbol{V} A)) = d(\ln(\text{constant}))$$

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

We can also obtain this relation by examining an infinitesimal element of streamtube:



Applying conservation of mass:

$$\rho VA = (\rho + d\rho)(V + dV)(A + dA)$$

Expanding:

$$\rho VA = (\rho V + Vd\rho + \rho dV + d\rho dV)(A + dA)$$

We can discard $d\rho dV$ as a higher order term. Expanded again and discarding other higher order terms:

$$\rho VA = \rho VA + VAd\rho + \rho AdV + \rho VdA$$

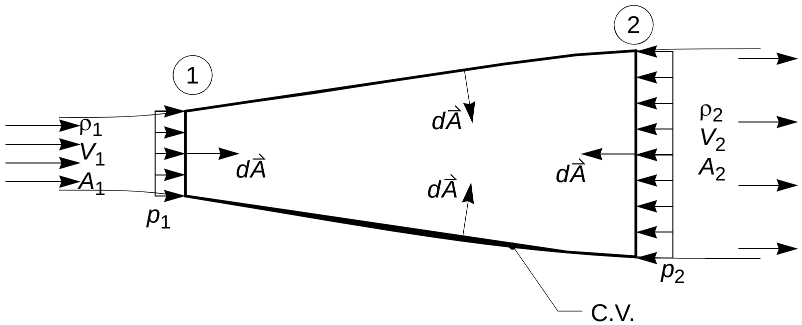
Which simplifies to:

steady, 1-D

$$\boxed{\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0}$$

3.2 Conservation of Momentum for 1-D Stream Tube

Going through the same exercise for momentum:



$$\Sigma\vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V}dV - \int_{CS} \vec{V}(\rho\vec{V}\cdot d\vec{A})$$

Specializing for the x-component of momentum in this one dimensional streamtube:

$$\Sigma F_x = \frac{d}{dt}\int_{CV} \rho VAdx - [\rho_1 V_1^2 A_1 - \rho_2 V_2^2 A_2]$$

We will consider the only forces acting on the control volume to be the pressure forces acting on the inlet and exit and the wall force (both pressure force and viscous drag). Thus, we will neglect body forces such as gravity or electromagnetic forces. We will break F_x into F_{xwall} and the pressure forces at inlet and exit:

$$F_{xwall} + p_1A_1 - p_2A_2 = \frac{d}{dt}\int_{CV} \rho VAdx - [\rho_1 V_1^2 A_1 - \rho_2 V_2^2 A_2]$$

Rearranging:

$$F_{xwall} = \frac{d}{dt}\int_{CV} \rho VAdx + A_2[p_2 + \rho_2 V_2^2] - A_1[p_1 + \rho_1 V_1^2]$$

If the flow is steady:

steady, 1-D

$$F_{xwall} = A_2[p_2 + \rho_2 V_2^2] - A_1[p_1 + \rho_1 V_1^2]$$

If we specialize further to a constant area flow ($A_1 = A_2 =$ constant):

$$\frac{F_{xwall}}{A} = [p_2 + \rho_2 V_2^2] - [p_1 + \rho_1 V_1^2]$$

If the flow is also inviscid (no friction), then the walls are incapable of exerting a force in the x-direction:

steady, 1-D, constant A, inviscid

$$p_2 + \rho_2 V_2^2 = p_1 + \rho_1 V_1^2$$

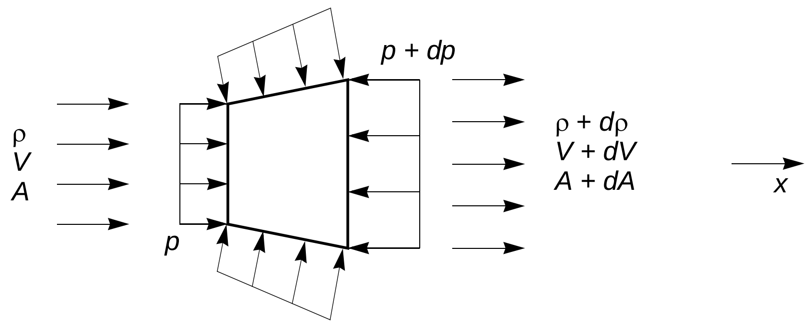
Note that this is *not* Bernoulli's equation (Bernoulli: $p_2 + \frac{1}{2}\rho_2 V_2^2 = \text{constant}$).

In general, we will be dealing with flows with area change and wall force:

steady, 1-D

$$F_{xwall} = A_2[p_2 + \rho_2 V_2^2] - A_1[p_1 + \rho_1 V_1^2]$$
$$= A_2 p_2 + V_2\dot{m} - A_1 p_1 - V_1\dot{m}$$

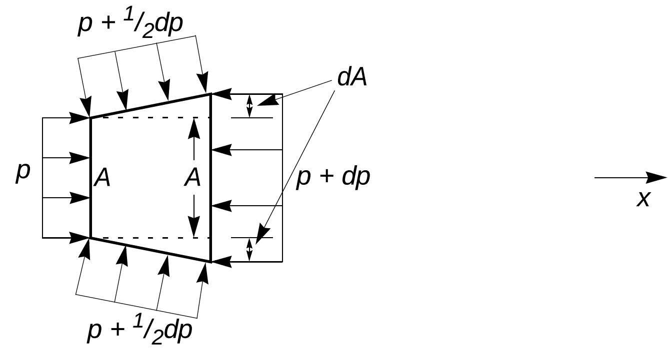
Again, we would like a differential version of the momentum equation as well. Examining a differential streamtube element:



We will assume that the flow is inviscid (no friction), so that the only force acting on the side walls is pressure. In the differential element as we have shown it here, the pressure on the side walls is assumed to vary linearly along the walls of the element (from p at the inlet to $p + dp$ at the exit). We can approximate this side wall pressure by the average of these two values:

$$p_{avg} = \frac{p + (p + dp)}{2} = p + \frac{dp}{2}$$

The component of this side wall force in the x-direction will be the product of this average pressure and the projection of the side wall area in the x-direction, which is simply dA.



$$F_{xwall} = p_{avg} dA = \left(p + \frac{dp}{2}\right) dA$$

Thus, the momentum equation becomes:

$$\left(p + \frac{dp}{2}\right) dA = (A + dA)(p + dp) + (V + dV)\dot{m} - Ap - V\dot{m}$$

Expanding and eliminating higher order terms (e.g., $dAdp$), we are left with:

steady, 1-D, inviscid

$$\boxed{\frac{dp}{\rho} + VdV = 0}$$

3.3 Conservation of Energy for 1-D Stream Tube

The conservation of energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV} e\rho dV = \dot{W}_{cv} + \dot{Q} + \int_{CS}\left(h + \frac{\vec{V}^2}{2}\right)\rho\vec{V}\cdot d\vec{A}$$

Applied to our one-dimensional, steady streamtube yields:

$$0 = \dot{Q} + \dot{W} + \left(h_1 + \frac{V_1^2}{2}\right)\dot{m} - \left(h_2 + \frac{V_2^2}{2}\right)\dot{m}$$

Dividing by $\dot{m}$:

$$0 = \frac{\dot{Q}}{\dot{m}} + \frac{\dot{W}}{\dot{m}} + \left(h_1 + \frac{V_1^2}{2}\right) - \left(h_2 + \frac{V_2^2}{2}\right)$$

The terms $\frac{\dot{Q}}{\dot{m}}$ and $\frac{\dot{W}}{\dot{m}}$ are the rate of heat addition [kJ/s] divided by the mass flow rate [kg/s]. Thus, these terms are simply the heat addition and work per unit mass [kJ/kg], denoted q and w, respectively:

steady, 1-D

$$\boxed{q + w + \left(h_1 + \frac{V_1^2}{2}\right) - \left(h_2 + \frac{V_2^2}{2}\right) = 0}$$

Note that we did not need to make any assumptions regarding friction, so this equation is valid even for viscous flow. In general, we will not be adding or extracting work via shafts, E-M, etc., so $w = 0$.

steady, 1-D

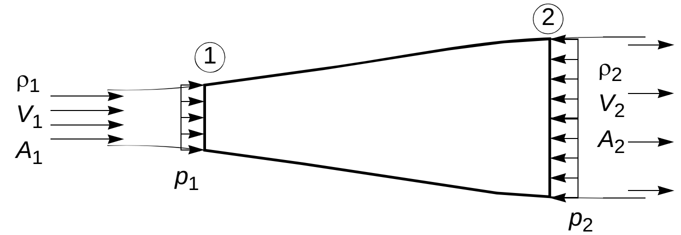
$$\boxed{q + \left(h_1 + \frac{V_1^2}{2}\right) - \left(h_2 + \frac{V_2^2}{2}\right) = 0}$$

We will also have need of a differential version of the energy equation. Following the same procedure we used for continuity, the energy equation can be written as:

steady, 1-D

$$\boxed{dh + VdV = dq}$$

Summarizing our results for a steady, one-dimensional streamtube:



	Integral/Control Volume	Differential
Continuity	$\rho_1 \boldsymbol{V}_1 A_1 = \rho_2 \boldsymbol{V}_2 A_2$	$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$
Momentum	$F_{xwall} = A_2[p_2 + \rho_2 V_2^2] - A_1[p_1 + \rho_1 V_1^2]$ $= A_2 p_2 + V_2\dot{m} - A_1 p_1 - V_1\dot{m}$	<u>Inviscid</u> $\frac{dp}{\rho} + VdV = 0$
Energy	$\left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \frac{V_1^2}{2}\right) = q$	$dh + VdV = dq$
	<u>Adiabatic</u> $\left(h + \frac{V^2}{2}\right) = \text{constant}$	<u>Adiabatic</u> $dh + VdV = 0$

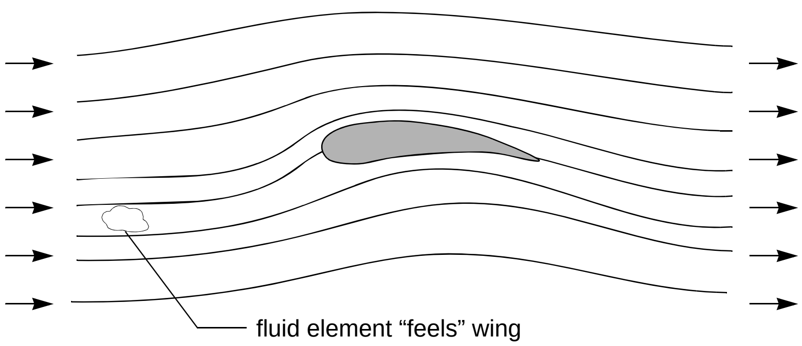
A final comment on the applicability of these equations: it is worth re-emphasizing that these equations can only be applied to *steady* flow.

An additional caveat must be issued regarding the differential equations: they are only applicable where the fluid properties (pressure, density, velocity) are *smooth* and *continuous*, that is, in regions of flow where the derivative is defined. At flow discontinuities (such as shock waves), the differential equations cannot be applied. The integral/control volume formulations, however, are always valid.

4. Speed of Sound

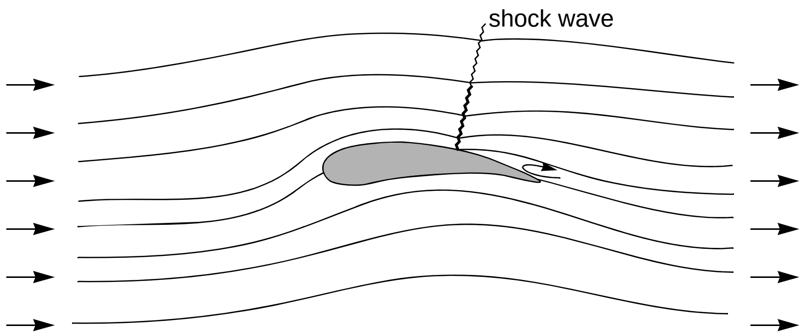
Now that we have a version of the conservation laws that can be applied to steady flows, we can begin to examine some specific flow problems. The speed of sound is a very important property in compressible flow. In fact, the essential characteristic that makes a flow "compressible" is when the flow velocity is comparable to the sound speed of the fluid (gas, liquid, etc.). In an incompressible flow, the flow was assumed to be so slow that the effects of a disturbance was felt everywhere instantly. This is why flow in the low speed (incompressible) regime is so smooth.

Consider a low speed flow over a wing:



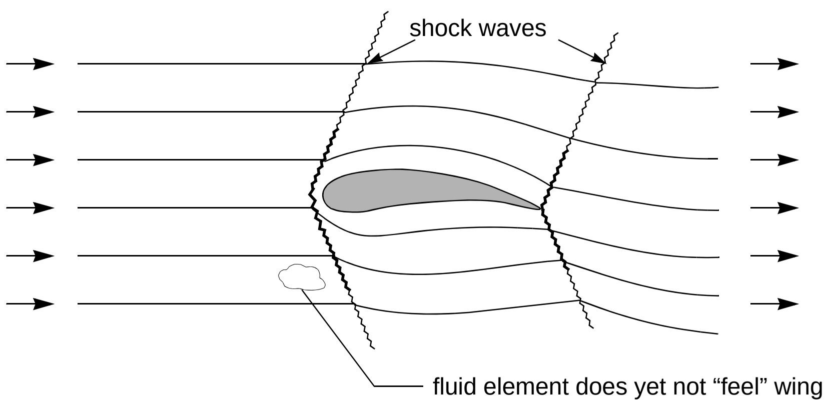
The element of air approaching the wing can "sense" the wing coming, so it has time to adjust and the resulting streamlines are smooth and continuous.

If we put the wing in a *high-speed subsonic flow*, the result is quite different:



Here the flow accelerates as it passes over the wing and reaches slightly supersonic speeds (even thought the approaching flow is subsonic). A sudden compression wave called a shock wave brings the flow to subsonic again. This phenomenon has a significant effect on the aerodynamic characteristics of the wing (lift, drag, flow separation, etc.).

If the same wing experiences *supersonic flow*, the flow is again different:



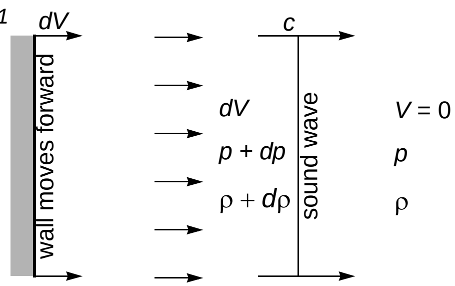
Now, the flow does not know of the wing's existence until is passes through the shock wave. Note that there is a second shock wave attached to the trailing edge of the wing which turns the flow back towards the free stream direction.

Note that we are already talking about subsonic and supersonic flow, without even knowing what the speed of sound is, but obviously the speed of sound will be an important parameter in compressible flow.

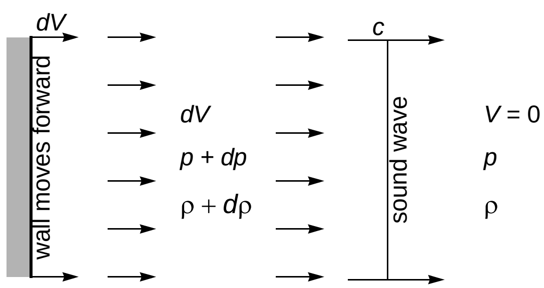
4.1 Derivation of the Speed of Sound

Sound is the mechanism by which a disturbance propagates through a compressible medium. Imagine a wall that bounds a compressible fluid is pushed outward with a velocity *dV*. This motion will result in the fluid near the wall also moving with velocity *dV*, and this disturbance will propagate outward as a sound wave. Fluid that has been "processed" by this wave will move at *dV*, fluid that the sound wave has not yet reached will still be at rest. We will denote this wave with wave speed *c*.

At time t_1

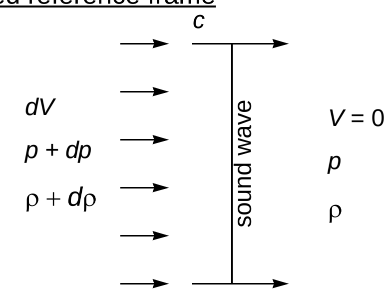


At time t_2

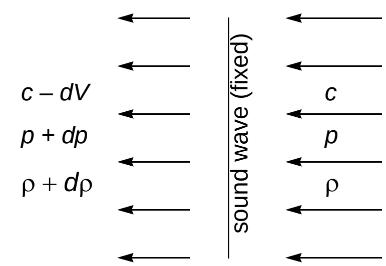


Can we analyze this flow with the tools we have developed so far? Not directly, because this is an *unsteady* flow that changes with time. To transform this picture to one in which the wave motion is steady, we can add a velocity *c* in the direction opposite to the direction of wave motion. This "Galilean" transformation is really just the same thing as observing the wave from a frame of reference that is moving at the same speed as the wave. If we are running along the wave at the same speed *c*, the wave appears at rest and now the flow that was quiescent appears to approach at speed *c*.

Viewed from fixed reference frame

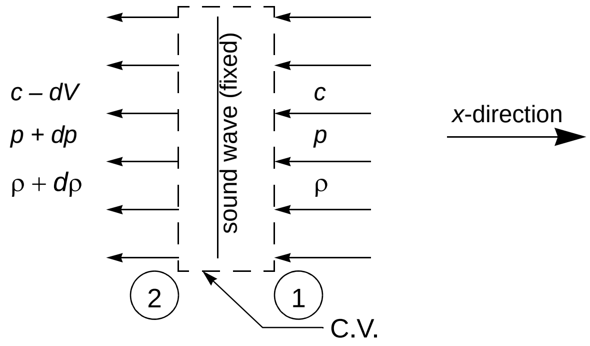


Viewed from moving reference frame



Note that thermodynamic parameters (*p*, ρ, etc.) are *independent* of the reference frame; only the velocity changes as we change reference frames.

Now we can draw a one-dimensional control volume around the steady flow passing though the fixed wave.



Here, we are assuming that the pressure and density increase by an infinitesimal amount dp and $d\rho$ as the wave passes.* We really do not know at this point that they do increase, but we will assume they do and then by examining the sign of dp and $d\rho$, we will be able to determine if our assumption was correct. Also note that the flow through the sound wave is constant area: $A_1 = A_2$.

Applying continuity

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

$$-\rho c = -(\rho + d\rho)(c - dv)$$

and discarding higher order terms, we get:

$$\rho dV = c d\rho$$

Since ρ, dV and c are all positive, $d\rho$ must also be positive. So, a sound wave that pushes *into* a medium is a wave that *increases the density* of the medium, which makes sense. Of course, we could have obtained this directly from the differential version of the continuity equation:

$$\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{dV}{V} = 0$$

and taking $dA = 0$ and $V = -c$.

This equation is not enough to solve for c, so we turn to momentum for constant area flow:

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2$$

$$p + dp + (\rho + d\rho)(c - dV)^2 = p + \rho c^2$$

Discarding higher order terms:

$$dp - 2c\rho dV + c^2 d\rho = 0$$

The ρdV term can be replaced with $c d\rho$ by using the continuity equation:

$$dp - 2c^2 d\rho + c^2 d\rho = 0$$

Solving for c:

$$c^2 = \frac{dp}{d\rho}$$

In other words, the speed of sound is related to the ratio of pressure change dp to density change $d\rho$ across the sound wave. Note that dp and $d\rho$ must have the same sign, so that the sound wave that is increasing density is also increasing pressure. This sound wave is a *compression wave*. We can repeat the analysis for a wave in which the wall pulls away from the medium (dV in the negative x-direction), and we will find that a *rarefaction wave* will propagate at the same sound speed into the medium that lowers density and pressure.

*If we assume that the changes across the wave are *finite* rather than infinitesimal, then we are considering a shock wave, which is discussed in greater detail in Chapter 7.

Since pressure is a function of two thermodynamic variables (e.g., $p(\rho,T)$), this derivative is more properly expressed as a partial derivative:

$$c^2 = \frac{\partial p}{\partial \rho}$$

How do we compute this partial derivative? There are some options: we could hold T constant (isothermal), or we could evaluate this derivative along some other thermodynamic path. Historically, it took a while before we understood how to evaluate this derivative (see Historical Note below), but we know now that sound waves are isentropic. So, we will evaluate this derivative along a path of constant entropy

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s\ =\ \mathrm{constant}}$$

For an isentropic process, pressure and density are related by $\frac{p}{\rho^\gamma} = \mathrm{constant}$, so we can substitute in $\rho^\gamma \cdot \mathrm{constant}$ for p when evaluating this derivative.

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s\ =\ \mathrm{constant}} = \frac{\mathrm{d}}{\mathrm{d}\rho}(\rho^\gamma \cdot \mathrm{constant})$$

$$= \gamma\rho^{\gamma-1}(\mathrm{constant})$$

$$= \frac{\gamma\rho^\gamma(\mathrm{constant})}{\rho}$$

We can now substitute p back in for $\rho^\gamma \cdot \mathrm{constant}$:

$$= \frac{\gamma p}{\rho}$$

Thus, c^2 is directly proportional to the ratio of pressure to density. For an ideal gas, this ratio is directly proportional to temperature $\frac{p}{\rho} = RT$, so

$$c^2 = \gamma RT$$

$$c = \sqrt{\gamma RT}$$

It is always a good idea to check the units on an important new result like this. Since R has units $\left[\frac{\mathrm{J}}{\mathrm{kg \cdot K}}\right]$ and γ is dimensionless, γRT has dimensions $\left[\frac{\mathrm{J}}{\mathrm{kg \cdot K}}\right][\mathrm{K}] = \left[\frac{\mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}^2}}{\mathrm{kg \cdot K}}\right][\mathrm{K}] = \left[\frac{\mathrm{m}^2}{\mathrm{s}^2}\right]$. So, c will have dimensions of m/s, as we would expect.

For air, the average molecular weight is MW = 28.8, so $R = \frac{\mathfrak{R}}{MW} = \frac{8314}{28.8}$ and $\gamma = 5/3 = 1.4$ for diatomic gases (oxygen and nitrogen). Thus, at room temperature (T = 300 K), $c = \sqrt{\gamma RT}$ = 348 m/s, or about 1250 km/hr.

The result that the speed of sound for an ideal gas is only a function of temperature may be surprising. This implies that as we increase or decrease the density of a gas, the sound speed remains constant if the temperature remains constant.* Does this make sense? Recall that temperature is a measure of the average kinetic energy of the random motion of the molecules of gas. Thus, $T \propto \frac{1}{2}mV_{avg}^2$, where V_{avg} is an average velocity of the molecular motion. So, the average velocity of the molecules is $V_{avg} \propto \sqrt{T}$. Since sound

*Of course, if we increase the pressure/density to a point where the ideal gas assumption is no longer valid, then the sound speed will not remain constant. For air at ambient temperature, these "real gas" effects do not become significant until hundreds of atmospheres of pressure.

is a disturbance propagated by the motion and collisions of the molecules, it makes sense that the speed at which sound propagates should depend on the speed of the molecules. As the gas is made more dense, the collisions between molecules becomes more frequent, but the molecules still move at the same speed if the temperature is the same, because the average distance between collisions is still very large compared to the volume occupied by the molecule. If you study the Kinetic Theory of Gases, you will find that the average molecular velocity and the speed of sound for an ideal gas are very closely related and numerically have a similar value (typically within 25% or so of each other).

Of course, this discussion applies only to ideal gases. As the gas density increases to very large values, we can no longer ignore the volume occupied by the molecules themselves, and "real gas" effects begin to appear. In liquids and solids, the molecules are in contact with their neighbors at all times, and for a crystalline material, all the molecules are chemically bonded to each other. Obviously, the equation of state for these materials are very different. Yet, the relation $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s\,=\,\text{constant}}$ is still valid, because we did not make any assumptions about the equation of state in deriving it. The challenge becomes evaluating the partial derivative for materials when the ideal gas equation of state does not apply.

For all materials, we can define an isentropic compressibility $K_s = \frac{1}{\rho}\left(\frac{d\rho}{dp}\right)_s$. K_s is the fractional change in density per unit change in pressure. Thus, K_s is a direct measure of how compressible a material is. Sound speed then becomes

$$c = \sqrt{\frac{1}{\rho K_s}}$$

For example, for different materials

	air	water	steel
K_s [ms^2/kg]	7 x 10^{-6}	4.55 x 10^{-10}	6.1 x 10^{-12}
ρ [kg/m^3]	1.2	1000	7900
c [m/s]	345	1480	4560

we see that water and steel are much more incompressible compared to air, and they have a correspondingly much larger sound speed.

4.1.1 Historical Note: The Speed of Sound


Newton (1642-1727)

Newton (1687) made the first attempt at calculating the speed of sound in his *Principia*. He had already measured the speed of sound by observing distant firings of cannons and comparing the difference in time between when he observed the flash of the cannon firing and when he heard the "bang." Experimentally, he obtained a value of 348 m/s. In calculating the speed of sound, Newton followed our analysis above, but when he got to the partial derivative $c^2 = \frac{\partial p}{\partial \rho}$, he assumed that a sound wave is *isothermal* (constant temperature). Newton probably reasoned that a sound wave is so fast, there is not time for the air to "heat up," so the temperature should be constant. Assuming constant temperature, $p \propto \rho$, the sound speed is

$$c = \sqrt{RT}$$

Which gives a value of 294 m/s, about 17% too low.

Theodore von Karman comments:


von Karman (1881-1963)

> Newton of course noticed the difference in figures obtained from theory and experiment. Then he followed a method familiar to graduate students, namely, he looked for some excuse to justify the discrepancy. First he remarked that the air was not clean; it always contains some suspended dust particles. He thought that the dust particles would account for a deviation of about 10 percent. Then he thought that moisture content would also act against compression. So he said these two effects together might be responsible for the 17 percent difference. Even very great men sometimes indulge in wishful thinking, which is perhaps a shortcoming of most research men. We must realize, however, that at this time thermodynamics was not known as a science.
>
> –Theodore von Karman, *Aerodynamics*, 1954.


Laplace (1749-1827)

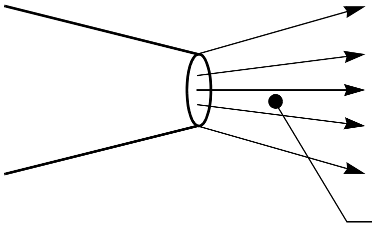
The correct calculation comes from Pierre Simon, Marquis de Laplace (1816), who realized Newton's mistake is that sound is too fast for heat transfer to occur, but the air does change temperature due to compression. Thus, sound is an adiabatic process. Also, since the change in flow properties and gradients are small, the sound wave should be reversible. A process that is adiabatic and reversible is *isentropic*. This is the assumption we used in our development, and the result matches experimental measurements of the sound velocity exactly.

4.2 Mach Number

Now that we have the sound speed, we can introduce an important parameter called Mach number, M. Mach number is defined as: "the ratio of the local flow speed to the local speed of sound."

$$M = \frac{|V|}{c}$$

Note the emphasis on *local*, meaning "at that point." If we examine a jet, for example:



the Mach number at point A is

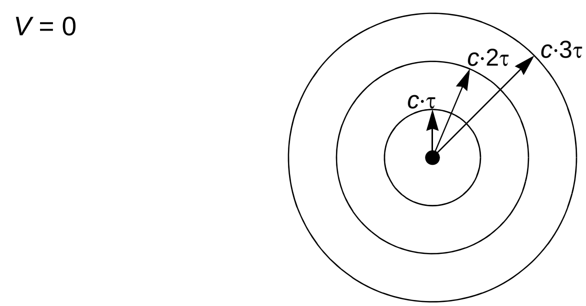
$$M_A = \frac{|V_A|}{c_A}$$

where V_A and c_A are the flow velocity and sound speed ***at the point A***. Note we do *not* use the ambient speed of sound or the speed of sound of the reservoir that drives the jet.

Note that you sometimes hear statements like, "The Concorde goes Mach 2." While this is a popular use of the term Mach number, it is not really the proper use. It would be more proper to say, "If we are on board the Concorde (i.e., if we are in the Concorde's reference frame), the air flow approaches us at Mach 2." This means that the air is approaching at twice the speed of sound of the air *at that altitude*. Thus, the actual speed (in m/s) that the Concorde is traveling depends on the temperature at that altitude.

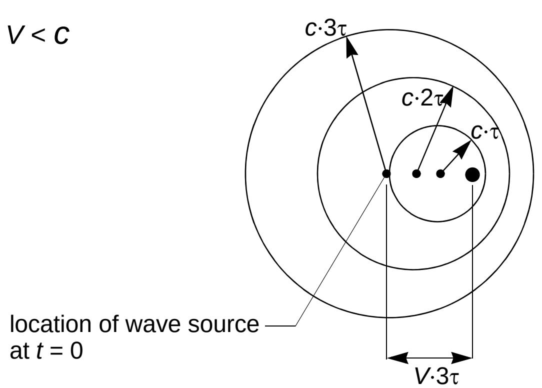
4.3 Subsonic *vs.* Supersonic Flow

Mach number is the single most important parameter to describe compressible flow, and there are profound differences between subsonic and supersonic flow. To illustrate the important difference between subsonic and supersonic flow, let us consider a small "beeper" that emits a "beep" after every time interval τ. If we hold this beeper in a quiescent fluid, the sound waves will radiate outward as concentric spheres with radius ct, where c is the speed of sound, and t is the time since the sound wave was generated. If we look at three such sound waves at an instant in time:



They propagate outward concentrically. This picture corresponds to the ripples set up if we throw a sequence of three rocks into the center of an initially still pond.

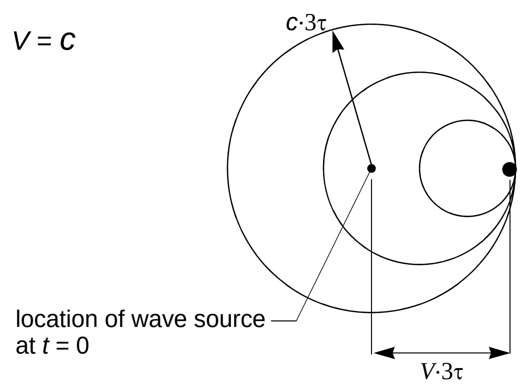
If the beeper now starts to move with a velocity V (that is still less than c), the sound waves emitted will still expand outward spherically, but they will no longer be centered on the beeper, which has now moved from the point where each wave was initially emitted:



Note that the sound waves are closer together in the direction of the beeper's motion, and farther apart in the direction opposite to the motion. To a stationary observer, the sound waves would come with a faster frequency as the beeper approached, and at a lower frequency as the beeper after the beeper passed and receded. This is what is responsible for the well-know "Doppler Shift" of sound: the frequency is shifted higher as a high speed object approaches, and shifted lower as it recedes.

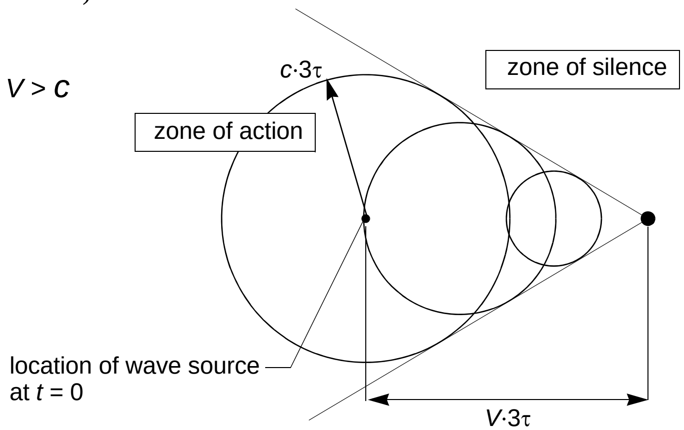
The exact same picture would apply if we held the beeper fixed and started a flow of fluid past it. This would correspond to dropping a sequence of stones at the same point into a flowing river (as opposed to a still pond).

If we now increase the speed of the of the source until its velocity equals the speed of sound ($V = c$), the sound waves will pile up in front of the source.

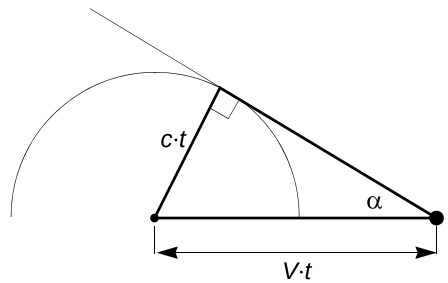


This picture also applies if the source is stationary and the fluid is flowing past at a velocity equal to c.

If we now increase the source to a velocity greater than c, the sound waves will form an envelope inside of which the source can be heard (the "zone of action").



Outside of this envelope, the source cannot be heard ("zone of silence"). Note from the geometry of the construction, we can find the half angle of the cone, α

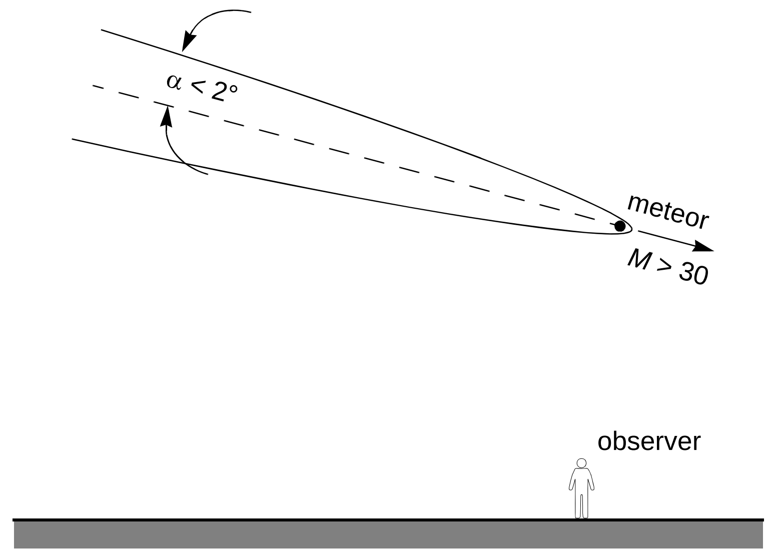


$$\sin\alpha = \frac{c}{V} = \frac{1}{\left(\frac{V}{c}\right)}$$

$$\alpha = \sin^{-1}\frac{1}{M}$$

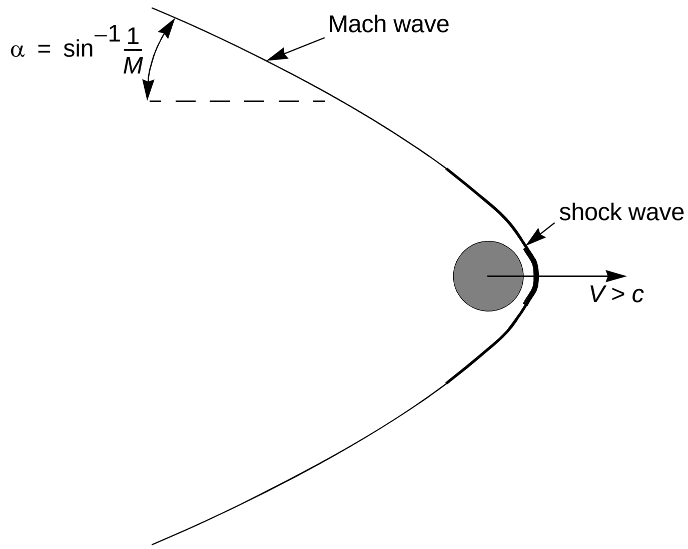
This angle is the envelope of disturbance waves emanating from a supersonic source (or, alternatively, a stationary source in a supersonic flow). This angle α is called the *Mach angle*. Note that as M increases, the Mach angle α decreases, meaning the region influenced by the source of sound waves is more narrowly confined.

To illustrate this point, consider a meteor streaking overhead. Meteors can be traveling anywhere from 12 to 70 km/s, corresponding to Mach numbers in the range $30 < M < 200$. This means that the Mach angle of noise generated by the meteor will be between $0.3° < \alpha < 2.0°$.

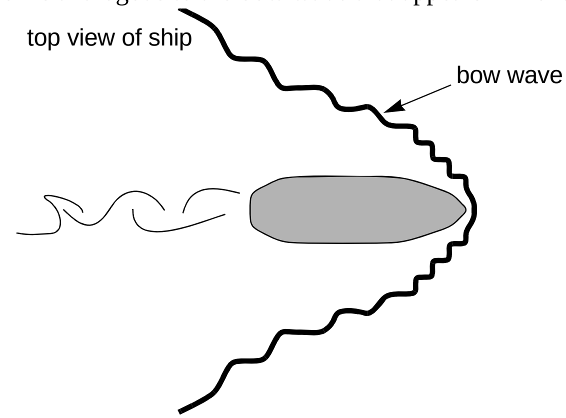


Thus, you would never hear a meteor as you see it streak overhead, but rather would have to wait several minutes for the sound to propagate down 50–80 km to reach you. By this time, the meteor is no longer visible.*

In reality, an object traveling at supersonic speeds creates a disturbance much stronger than sound called a *shock wave*. Although the shock wave propagates faster than the speed of sound, it weakens as it propagates away from the supersonic body and eventually decays to a sound wave at the Mach angle.



This bow shock is analogous to the bow wave that appears in front of ships.*



We have not yet treated the flow across a shock wave (subject of Chapter 7), so we cannot yet discuss these flows in quantitative detail. But these considerations clearly illustrate the profound differences between subsonic and supersonic flow. In fact, the concept of Mach number is so fundamental to compressible fluid flow, we will use it in the formulation of our developments for the different types of flow.

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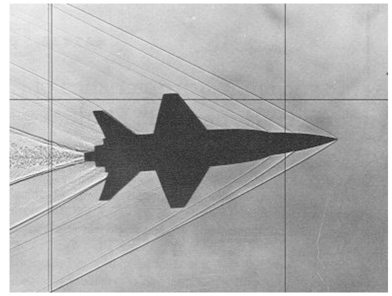
*Surprisingly, some observers *do* report hearing noise simultaneously with the observation of meteors, in contradiction to everything we know about sound propagation! This so-called "electrophonic" noise has only recently been recorded by scientists. These noises are believed to result from highly ionized gas in the meteor's trail generating radio waves, which propagate at the speed of light. Interaction of these radio waves with objects on the ground make audible "popping" noises effectively simultaneously with the meteor, although this theory is still highly speculative. See: Zgrablic, G. et al., "Instrumental recording of electrophonic sounds from Leonid fireballs," *Journal of Geophysical Research*, Vol. 107, No. A7, 2002, pp. SIA11-1-9.

*This analogy between bow shocks in supersonic flow and bow waves on ships is a limited analogy. For a ship in deep water, the surface waves to not propagate at a single velocity as in air, but rather over a spectrum of wave speeds that depends on wave length. The spectrum of waves generated by a ship interact in such a way so that the bow wave is always near an angle of 19.5°, regardless of the ship's speed or size! The resulting bow wave and wake is called the *Kelvin ship wave pattern*. This is obviously very different from supersonic aerodynamics, in which the Mach angle is directly linked to the speed of the object generating the bow shock.

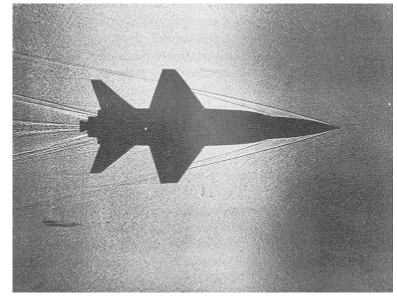
4.4 Picture Gallery

Andy Green reaches the sound barrier in the jet-powered car “Thrust SSC”. Black Rock Desert, Nevada, 1997.

“Thrust SSC” exceeds the speed of sound. Note Mach lines streaming from surface of vehicle.



Mach 3.5



Mach 6.0

Mach lines visible from a model of the X-15 rocket plane fired in an “aeroballistic” range. Note Mach angle decreases as Mach number increases.

Bow waves on the surface of water, created by the motion of bodies small and large.

5. Isentropic One-Dimensional Flow

We now turn our attention to a specific type of flow: isentropic flow. More specifically: one-dimensional, steady, isentropic flow. It is important to understand what each one of these terms mean:

- *one-dimensional*: slow variations in pipe or stream tube cross-sectional area.
- *steady*: not changing in time.
- *isentropic*: adiabatic and reversible.

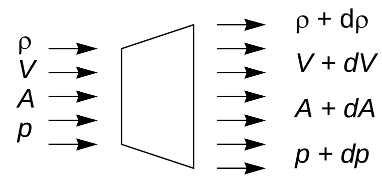
Adiabatic, in turn, means that there is no heat transfer to or from the flow, and reversible means that there can be no dissipation mechanism such as friction. Note that "no friction" means zero viscosity, $\mu = 0$, so that the Reynolds number $Re = \frac{\rho V L}{\mu}$ is undefined. In reality, viscous effects are always present in flows, but are usually confined to a narrow region along the wall called the boundary layer. Outside this region, flow can be very nearly isentropic. A further implication of isentropic is that the flow must be *smooth*, without discontinuities such as shock waves.

It is very important to remember that each of these terms refers to a limitation on where we can apply the relations we will derive in this chapter. Despite all of these restrictions, the relations of isentropic flow are extremely useful in modeling a number of real flows encountered in engineering.

5.1 Effect of Area Change

Just as in mechanics, where we often had to use impulse-momentum and work-energy techniques to solve various parts of a given problem, here we will also use combinations of the continuity, momentum, and energy equations to "solve" isentropic flow.

Let us begin by considering a differential stream tube element



and examine continuity: $\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{dV}{V} = 0$

and momentum: $\frac{dp}{\rho} + V dV = 0$ and try to get a feel for how the flow variables change as they go through this element of stream tube.

If we eliminate density by solving for density in the momentum equation:

$$\frac{1}{\rho} = -\frac{V dV}{dp}$$

and substituting back into continuity:

$$\frac{dA}{A} - V dV \frac{d\rho}{dp} + \frac{dV}{V} = 0$$

or
$$\frac{dA}{A} - \left(V^2 \frac{d\rho}{dp} - 1\right) \frac{dV}{V} = 0$$

Note the appearance of $d\rho/dp$. Since we are examining an isentropic flow, pressure and density are related by: $\frac{p}{\rho^\gamma} = \text{constant}$. Thus, we can evaluate this derivative. However, we note that we have already done this derivative in Section 4.1, and the result is the reciprocal of sound speed squared:

$$\left(\frac{\partial \rho}{\partial p}\right)_{s\,=\,\text{constant}} = \frac{1}{c^2}$$

Thus, we can replace $d\rho/dp$ by $1/c^2$:

$$\frac{dA}{A} - \left(\frac{V^2}{c^2} - 1\right)\frac{dV}{V} = 0$$

What is V/c? *Mach number, M*! Thus:

$$\frac{dA}{A} = (M^2 - 1)\frac{dV}{V}$$

or
$$\frac{dV}{V} = \frac{1}{M^2 - 1}\frac{dA}{A}$$

This expression is an extremely important result that has profound implications for compressible flow.

To interpret this expression, let us consider the diverging element of stream tube shown above, and let us assume the flow is subsonic: $M < 1$. Thus, the $\frac{1}{M^2 - 1}$ term is negative, and dA is positive (increasing area). Their product is negative, so $dV/V < 0$.

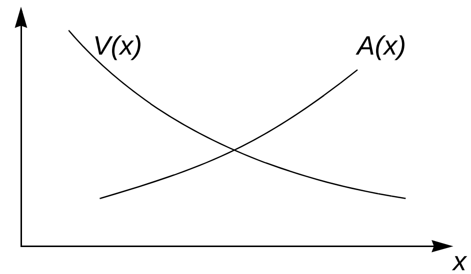
$$\frac{dV}{V} \propto -\frac{dA}{A}$$

Thus, velocity *decreases* when the area *increases* in *subsonic* flow.

If you are uncomfortable working with differentials, we can divide both sides by dx:

$$\frac{1}{V}\frac{dV}{dx} = \frac{1}{M^2 - 1}\frac{1}{A}\frac{dA}{dx}$$

and interpret the results based on the slope of the area profile of the stream tube:



Note that the velocity curve must have the opposite slope of the area curve.

If we examine a converging flow, then $dA < 0$, and for subsonic flow, dV will be positive. This means that subsonic flow accelerates with converging area. This is why a nozzle (device to accelerate flow) decreases the area of the flow in order to accelerate it.

If we now consider supersonic flow ($M > 1$), the $\frac{1}{M^2 - 1}$ term is positive, and therefore dV and dA will have the same sign.

$$\frac{dV}{V} \propto \frac{dA}{A}$$

A supersonic flow encountering an increase in area will accelerate. This is why a rocket nozzle has a diverging, bell-shaped area profile: the flow in the nozzle is supersonic and accelerates as the area increases.

Now we have a feel for how velocity responds to area change; what happens to pressure and density? From the momentum equation

$$\frac{dp}{\rho} + V\,dV = 0$$

We see that $\frac{dp}{\rho} = -VdV$. So, dp and dV will always have opposite signs. An increase in velocity is always associated with a decrease in pressure, and vice versa, for both subsonic and supersonic flow. We can replace dV in the momentum equation with our expression $\frac{dV}{V} = \frac{1}{M^2 - 1}\frac{dA}{A}$. Thus,

$$\frac{dp}{\rho V^2} = \frac{1}{1 - M^2}\frac{dA}{A}$$

Thus, pressure responds to area change in the opposite way as velocity. For density, we can multiply and divide this expression by $d\rho$.

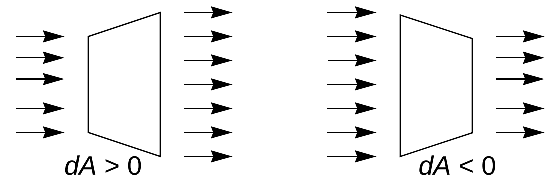
$$\frac{d\rho}{\rho V^2}\frac{dp}{d\rho} = \frac{1}{1 - M^2}\frac{dA}{A}$$

$$\frac{d\rho}{\rho}\frac{c^2}{V^2} = \frac{1}{1 - M^2}\frac{dA}{A}$$

$$\frac{d\rho}{\rho} = \frac{M^2}{1 - M^2}\frac{dA}{A}$$

Thus, density responds to area change similarly as pressure.

We can summarize these results:



	diverging flow	converging flow
subsonic flow ($M < 1$)	$dV < 0$ (decelerates)	$dV > 0$ (accelerates)
	$dp > 0$ (pressure increases)	$dp < 0$ (pressure decreases)
	$d\rho > 0$ (density increases)	$d\rho < 0$ (density decreases)
supersonic flow ($M > 1$)	$dV > 0$ (accelerates)	$dV < 0$ (decelerates)
	$dp < 0$ (pressure decreases)	$dp > 0$ (pressure increases)
	$d\rho < 0$ (density decreases)	$d\rho > 0$ (density increases)

Note that pressure, velocity, and density vary in the opposite way when we compare subsonic and supersonic flow. In other words, area change produces the *opposite effects* in subsonic and supersonic flow.

It is important that these trends be thoroughly understood and mastered before proceeding in our study of compressible flow. It is essential that a conceptual feel for how flow responds to area change be developed in order for the continued development of compressible flow to make sense.

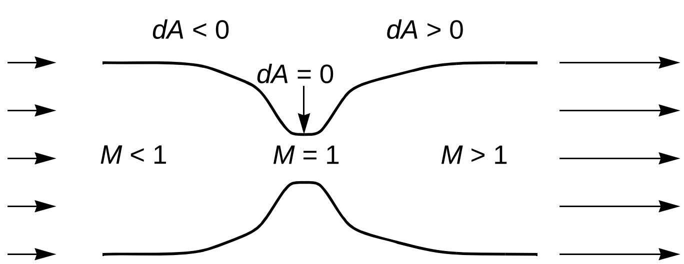
An obvious question that arises at this point is: *What happens at Mach 1*? Examining the relation

$$\frac{dV}{V} = \frac{1}{M^2 - 1}\frac{dA}{A}$$

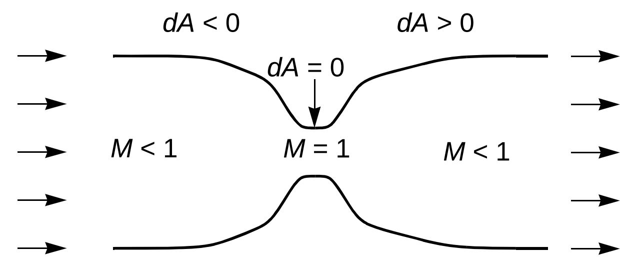
We can see that the only way that $M = 1$ can be permitted without making the dV become undefined is for dA to also be zero. In other words, sonic flow ($M = 1$) is only permitted where $dA = 0$, i.e., in a constant area section of flow.

It is tempting to pose what would happen if we forced sonic flow into a converging or diverging channel, which is something we can certainly do experimentally. But since this case is not mathematically permitted by our relations, this case must violate our assumptions, specifically, our assumption of steady flow. The result of such an experiment can only be dealt with using an unsteady flow model, such as discussed in Chapter 16. For the flow considered in this chapter, that is, *steady one-dimensional isentropic flow,* ***sonic flow can only occur in a constant area section***.

A subsonic flow cannot be accelerated to supersonic flow by area convergence alone, since when it reaches sonic, the area of the flow must be constant. But, a subsonic flow may be accelerated to sonic in a converging area steamtube, and then may go supersonic by a diverging stream tube following the converging one.



The flow at the constant area section ($dA = 0$) or "throat" that joins the converging and diverging sections would have to be exactly sonic ($M = 1$). The flow may also decelerate back to subsonic and remain subsonic in the diverging section right now.

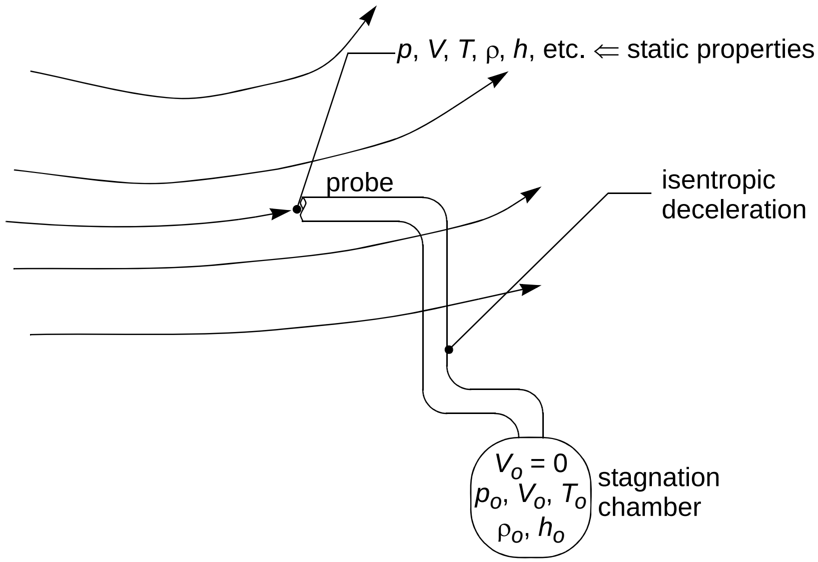


Thus, there are two possible solutions to the problem of subsonic flow accelerated to sonic at the throat of a converging/diverging streamtube. We cannot yet say which case will happen, and a satisfactory treatment of this problem will have to wait until Chapter 8. For now, we will say that the flow is indeterminate; either case could happen.

5.2 Stagnation Conditions

Imagine that we have a probe we can insert into a flow and sample the flow at a point. This will be a very special probe that can decelerate the flow isentropically to a sample chamber where the velocity of the flow is zero.* When the flow is brought to zero velocity, it has been *stagnated*, and we will refer to the fluid properties at that state as *stagnation conditions*. Stagnation conditions (also referred to as "total conditions") are denoted with a subscript "o". For example, p_o, T_o, h_o are the stagnation pressure, stagnation temperature, and stagnation enthalpy, respectively. These are not the properties that the flow actually *feels*. What the flow feels are the static properties: p, T, h, etc. Static properties are what a barometer, thermometer, or enthalpy meter would measure if they were carried along in the flow. Stagnation conditions, on the other hand, are a reference state of what the properties would be if the flow was brought to rest.

*In practice, such a probe is called a "pitot probe" and extremely difficult to construct in order to isentropically decelerate a flow, esp. for supersonic flow. We will explore how to decelerate a supersonic flow isentropically further in Chapter 9.

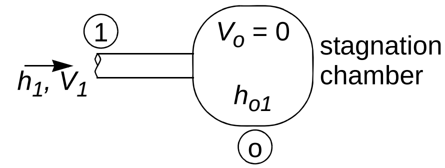


Note that static thermodynamic properties (p, ρ, etc.) are *independent* of the reference frame. The pressure at a point in the flow is the same, regardless of whether you are moving along with the flow, stationary with the flow going past you, or even if you are moving in the opposite direction. Stagnation properties, on the other hand, *are dependent on the reference frame*. For an observer moving with the flow velocity, the static pressure at that point is the stagnation pressure, since from the observer's point of view the flow is already at rest (stagnated). But for a stationary observer watching the fluid sweep by, the stagnation pressure will be greater than the static pressure.

If we now look at the energy equation for adiabatic flow

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = \text{constant}$$

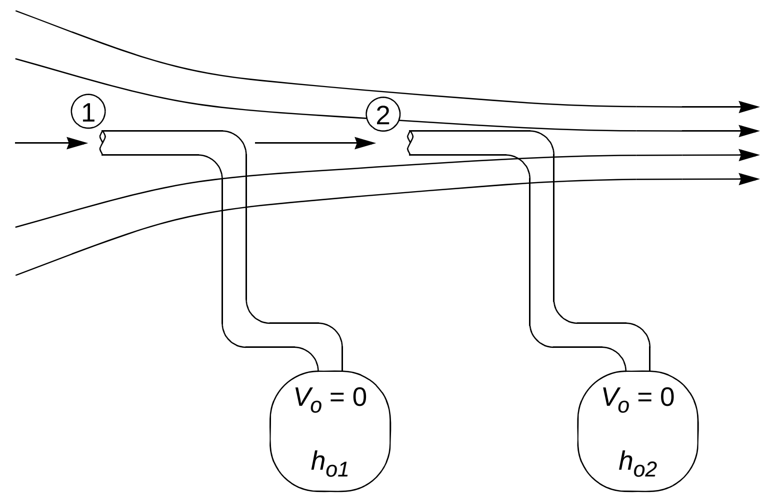
and apply it to the isentropic (and therefore adiabatic) flow in our stagnation probe



$$h_1 + \frac{V_1^2}{2} = h_{o1} + \cancelto{0}{\frac{V_o^2}{2}}$$

$$h_{o1} = h_1 + \frac{V_1^2}{2}$$

If we have a second stagnation probe somewhere else in the same adiabatic, one-dimensional flow, say at station 2



$$h_{o2} = h_2 + \frac{V_2^2}{2}$$

Because $h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$, therefore $h_{o1} = h_{o2}$. In other words, another expression of the energy equation for an adiabatic flow is:

$$h_o = \text{constant}$$

Since the main flow is adiabatic as well as the flow used to decelerate our "sample" of the flow into our stagnation probe, the energy equation applies everywhere, and whenever we decelerate the flow to zero velocity, we will obtain the *same* stagnation enthalpy. Thus, stagnation enthalpy becomes a useful reference state, even if the flow never decelerates to zero velocity.

For a calorically perfect gas, $h = c_pT$, where c_p is a constant. Therefore, $h_o = c_pT_o$, and another way to express conservation of energy is:

$$T_o = \text{constant}$$

If we want to relate static temperature T with stagnation temperature T_o, we use the definition of stagnation enthalpy:

$$h_o = h + \frac{V^2}{2}$$

$$c_pT_o = c_pT + \frac{V^2}{2}$$

Dividing by c_pT:

$$\frac{T_o}{T} = 1 + \frac{V^2}{2c_pT}$$

Whenever we see velocity V, we can always replace it with Mc, where M is Mach number and c is sound speed.

$$\frac{T_o}{T} = 1 + \frac{M^2c^2}{2c_pT}$$

Knowing that $c = \sqrt{\gamma RT}$, and the identities $R = c_p - c_v$ and $\gamma = c_p/c_v$, we can rearrange to yield

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}M^2$$

The equation relates the stagnation temperature to the static temperature. Recall that stagnation enthalpy is related to static enthalpy by the velocity at the static conditions: $h_o = h + \frac{V^2}{2}$. This relation for temperature is analogous, only now it is Mach number M that relates these two parameters. This expression will be one of the most common formulas used in compressible fluid dynamics. Note that the ratio $\frac{T_o}{T}$ is always greater than 1. This means that stagnation temperature is always greater that static temperature. When ever a flow is decelerated, its temperature will increase, and the maximum temperature occurs at stagnation conditions (zero velocity). Also note that this relation is valid for adiabatic flow, not just isentropic flow. Thus, it will have much wider applicability than just isentropic flow.

Since the flow in our stagnation probe is decelerated isentropically to zero velocity, we know how stagnation pressure and stagnation density are related to their static values. From the thermodynamic relations for an isentropic process:

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

and knowing that $\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}M^2$, we can conclude

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma - 1}} \qquad \frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma - 1}}$$

These equations give the relation between stagnation conditions (temperature, pressure, density) and static conditions. We see they are related by *Mach number.*

To summarize, the energy equation for an adiabatic flow is equivalent to:

T_o = constant

h_o = constant

For a flow that is adiabatic and reversible (i.e., isentropic):

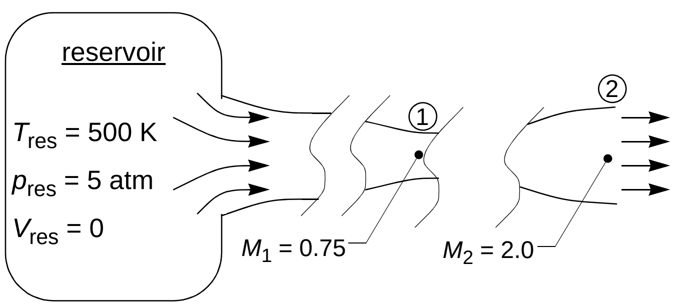
p_o = constant

ρ_o = constant

5.2.1 Numerical Example: Stagnation Conditions

We can illustrate the concept of stagnation properties and how we use them as reference states in a numerical example.

Problem: Consider a large, quiescent reservoir of high-pressure (5 atm), high-temperature (500 K) air that feeds a duct.



We can assume the flow is isentropic. At some point later in the duct (Point 1), we are told the flow is at Mach 0.75. At an even later point, the flow is at Mach 2.0. What are the pressure and temperature at those points?

Solution: Since the flow is adiabatic, the energy equation tells us that T_o = constant. What is T_o? T_o is the temperature of the flow when it is brought adiabatically to rest. The flow in the reservoir is already at rest, and it is connected to the rest of the flow in the duct by an adiabatic flow. Thus, the reservoir is already at the stagnation conditions! $T_o = T_{res}$. Since this stagnation conditions applies to the entire flow, it applies at point 1:

$$\frac{T_o}{T_1} = 1 + \frac{\gamma - 1}{2}M_1^2$$

Solving for T_1:

$$T_1 = \frac{T_o}{1 + \frac{\gamma - 1}{2}M_1^2}$$

Thus, the temperature at Point 1 is:

$$T_1 = \frac{500}{1 + \frac{1.4 - 1}{2}0.75^2} = 449 \text{ K}$$

Note that the static temperature dropped by about 50 °C as the flow accelerated from rest to Mach 0.75. Likewise, at Point 2:

$$T_2 = \frac{T_o}{1 + \frac{\gamma - 1}{2}M_2^2} = \frac{500}{1 + \frac{1.4 - 1}{2}2.0^2} = 278 \text{ K}$$

The static temperature drops to an even lower temperature (278 K) as the flow accelerates to Mach 2, but the stagnation temperature remains constant.

In order to find pressure, the fact that the flow is isentropic allows us to use the stagnation pressure as a reference state that is constant throughout the flow, which equals its value in the reservoir ($p_o = p_{res} = 5$ atm).

$$\frac{p_o}{p_1} = \left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{\frac{\gamma}{\gamma - 1}}$$

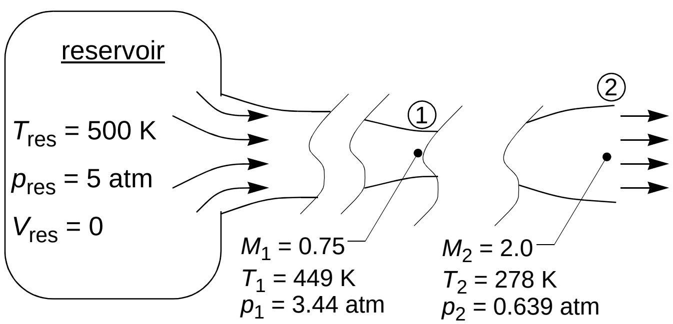
Solving for p_1:

$$p_1 = \frac{p_o}{\left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{\frac{\gamma}{\gamma - 1}}} = \frac{5\text{ atm}}{\left(1 + \frac{1.4 - 1}{2}0.75^2\right)^{\frac{\gamma}{\gamma - 1}}} = 3.44\text{ atm}$$

And p_2:

$$p_2 = \frac{p_o}{\left(1 + \frac{\gamma - 1}{2}M_2^2\right)^{\frac{\gamma}{\gamma - 1}}} = \frac{5\text{ atm}}{\left(1 + \frac{1.4 - 1}{2}2^2\right)^{\frac{\gamma}{\gamma - 1}}} = 0.639\text{ atm}$$

It is always good to summarize the results in tabular form, so that we can get a feel for the flow:



Note that as the flow accelerates from zero velocity (stagnation) to Mach 0.75 to Mach 2, the temperature and pressure decrease. This makes sense: the high upstream pressure is driving the flow, and pressure must drop as the flow accelerates. Further, the thermal energy of the gas is being converted into kinetic energy, so the temperature will drop as well.

We could also find velocity at each point, since we know temperature, and $c = \sqrt{\gamma RT}$ and $V = Mc$.

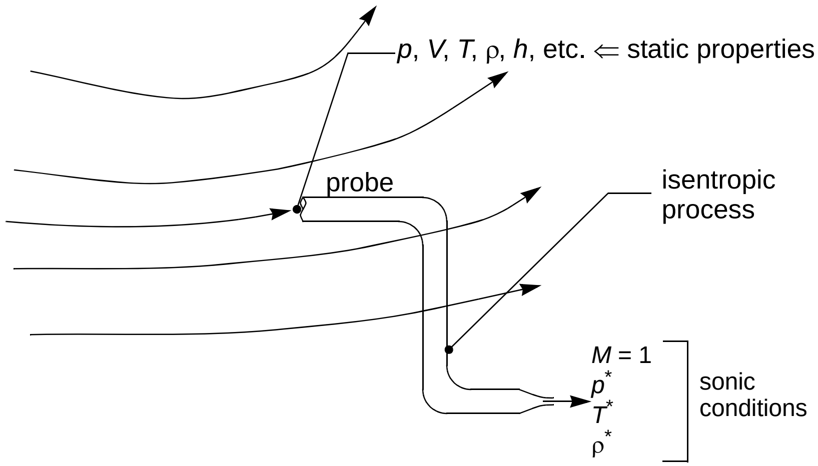
Comment: In this example, the stagnation conditions were given to us as the reservoir that feeds flow into the duct. In many problems, we will not be given the reservoir state (in fact, *there may not even be a reservoir*). However, as long as we know the Mach number and thermodynamic state at one point in the flow, we can find the stagnation condition that the flow would have if decelerated isentropically. This stagnation conditions becomes a reference state that we can use to help find the pressure, temperature, Mach number, etc. at other points in the flow. This is a general problem solving methodology that we will use throughout this course.

5.3 Sonic Conditions

Similar to stagnation conditions, the *sonic condition* is the state of a flow if brought isentropically to Mach 1. Again imagine a probe that can sample the flow and bring it isentropically to $M = 1$. The conditions at that state are called "sonic" or "critical" conditions, are denoted with "*" superscript. Since the relations above apply to all isentropic flow, we can apply them to determine the flow conditions of a flow brought to Mach 1. For temperature, we know from the prior section that

$$T_1 = \frac{T_o}{1 + \frac{\gamma - 1}{2}M_1^2}$$

If we take $M_1 = 1$, then $T_1 = T^*$. Thus,



$$T^* = \frac{T_o}{1+\frac{\gamma-1}{2}} = \frac{2}{\gamma+1}T_o$$

$$\frac{T^*}{T_o} = \frac{2}{\gamma+1}$$

Likewise, for pressure and density:

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

For air ($\gamma = 1.4$), the values are:

$$\frac{T^*}{T_o} = 0.833 \qquad \frac{p^*}{p_o} = 0.5283 \qquad \frac{\rho^*}{\rho_o} = 0.6339$$

These values for the ratio of sonic temperature to stagnation temperature and for sonic pressure to stagnation pressure appear repeatedly in compressible flow, and their numerical values are worth committing to memory.

We can also develop an expression for c^*.

$$\frac{c^*}{c_o} = \sqrt{\frac{T^*}{T_o}} = 0.912 \text{ (for } \gamma = 1.4)$$

Note: In some books on compressible fluid dynamics (Saad's *Compressible Fluid Flow*, for example), you will see the term M^*. You might think that this means $M^* = \frac{V^*}{c^*}$. Of course, at sonic conditions, $V = c$ by definition, so M^* would always be unity. Instead, these books use M^* to denote $M^* = \frac{V}{c^*}$, that is, the local velocity normalized by the speed of sound if the flow were brought isentropically to sonic. Recall that since c depends on temperature, the sonic speed will in general be different at sonic conditions than at the location flow conditions. Understandably, you may find this very confusing, so we will not use the expression M^* in these notes.

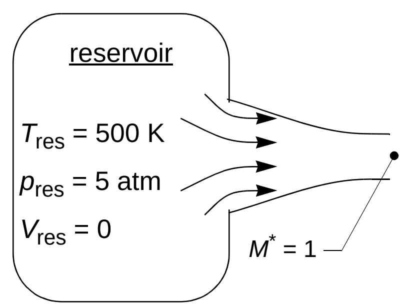
Sonic conditions, like stagnation conditions, are constant in an isentropic flow. Thus, they are another reference state we can use in solving problems.

5.3.1 Numerical Example: Sonic Conditions

Returning to the reservoir in Section 5.2.1, the conditions if the isentropic flow were brought to sonic would be:

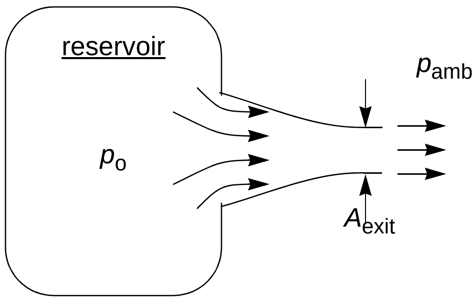
$$\frac{T^*}{T_o} = 0.833 \Rightarrow T^* = 417\text{ K}$$

$$\frac{p^*}{p_o} = 0.5283 \Rightarrow p^* = 2.64\text{ atm}$$



5.4 Choking

Closely related to the sonic condition is the concept of "choking." To introduce choking, consider for a minute a reservoir of gas at conditions T_o, p_o. For this discussion, we will assume the reservoir is very large and the reservoir conditions can be assumed constant and fixed. The reservoir is surrounded by ambient pressure, p_{amb}.



If the ambient pressure equals the reservoir pressure ($p_o = p_{amb}$), no flow occurs. If the ambient pressure is lowered ($p_o > p_{amb}$), then flow starts to pass though the exit area. If the ambient pressure is lowered further, the mass flow rate through that area increases, because the gas is more strongly "sucked" out. Below a certain value of ambient pressure, however, the mass flow rate is maximized, and lowering the ambient pressure further does not increase the mass flow rate.

For example, the reservoir could be a SCUBA tank of high pressure air that we are draining because we want to take it on our vacation, and the airlines do not permit pressurized tanks to fly. If the tank is taking too long to drain, we might become impatient. In order to increase the mass flow rate though the valve, we might be tempted to hook up the tank to a vacuum pump to "suck" the air out faster. But we will find that, even if we hook up the tank to a very powerful vacuum pump that can suck a large mass flow down to very low pressures, it will *not* increase the rate of mass flow leaving the tank. That is because the flow is *choked* at the smallest constriction point in the valve of the tank.

To visualize choking, imagine a crowded room full of people, all trying to get through the door onto the patio outside. If the patio is also crowded, then the density of people outside will limit how fast people inside can exit the room. But even if the patio is empty and there is plenty of room outside, there is still a limit to how fast people can jam and squeeze their way through the door. When this happens, we say the exit flow is *choked* and is independent of the conditions outside the room.

To see when this choking occurs, let us examine the mass flow from continuity:

$$\dot{m} = \rho V A$$

This expression applies everywhere in the flow in the exit duct, including the exit area. We will express the mass flow on a per-unit-area basis:

$$\frac{\dot{m}}{A} = \rho V$$

In order to understand when choking occurs, we would like to put the mass flow per unit area in terms of the reservoir conditions (which we will assume are fixed) and the Mach number at that area. We can express ρ and V in terms of p, T and the Mach number at p and T as follows:

$$\frac{\dot{m}}{A} = \left(\frac{p}{RT}\right)(M \cdot c)$$

Using our expression for the speed of sound:

$$\frac{\dot{m}}{A} = \left(\frac{p}{RT}\right)(M \cdot \sqrt{\gamma RT})$$

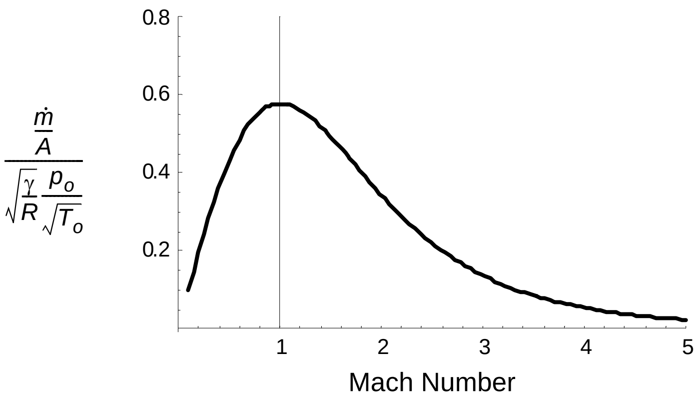
Expressing p and T in terms of the stagnation conditions and the local Mach number:

$$p = \frac{p_o}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}} \qquad T = \frac{T_o}{1+\frac{\gamma-1}{2}M^2}$$

Our expression for mass flow per unit area can be written as:

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}}\frac{p_o}{\sqrt{T_o}}\frac{M}{\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

This expression gives the mass flow per unit area $\frac{\dot{m}}{A}$ in terms of the fixed stagnation conditions and the Mach number at that area. We can plot this expression as a function of Mach number; note we actually plot $\frac{\dot{m}}{A}$ nondimensionalized by $\sqrt{\frac{\gamma}{R}}\frac{p_o}{\sqrt{T_o}}$, which is a constant for fixed stagnation conditions. We see that the maximum mass flow occurs at $M = 1$. We can prove this by equating the derivative $\frac{d}{dM}\left(\frac{\dot{m}}{A}\right)$ to zero and solving for M. This will verify that $M = 1$ is an extremum, and from the plot it is obviously the maximum of the curve.



This result means that the maximum mass flow we can pass through a given area occurs when the flow is at sonic. You might be tempted to think, "But wouldn't increasing the flow Mach number to, say, Mach 3 increase the mass flow?" The answer is: ***No!*** While increasing the flow to Mach 3 would increase the velocity, recall we are dealing with a compressible flow. Once supersonic, density decreases faster than velocity increases; that is why supersonic flow must accelerate in a diverging channel to maintain continuity. At Mach 3, the lower density would more than offset the increased velocity, lowering the net mass flow rate. Again, you might think, "Can't we increase the density of the Mach 3 flow?" Again, the answer is: *No*, because the stagnation conditions are fixed, the density and velocity are linked by the relations of isentropic flow. The maximum product of ρV, as we have just shown, occurs at $M = 1$. In other words, if you are trying to maximize the mass flow you can pass through a given area, you want the flow at that area to be sonic.

While this result may appear counterintuitive, it actually makes a great deal of sense that maximum mass flow occurs at $M = 1$. As we lower the ambient pressure, this fact is communicated to the flow by acoustic waves. However, when the flow becomes sonic, acoustic waves can no longer propagate upstream to increase the mass flow rate. Thus, the mass flow rate becomes fixed and independent of the ambient pressure; we call this condition *choking*.

What is the maximum mass flow, for a fixed stagnation condition. For our expression for $\frac{\dot{m}}{A}$ evaluated at $M = 1$:

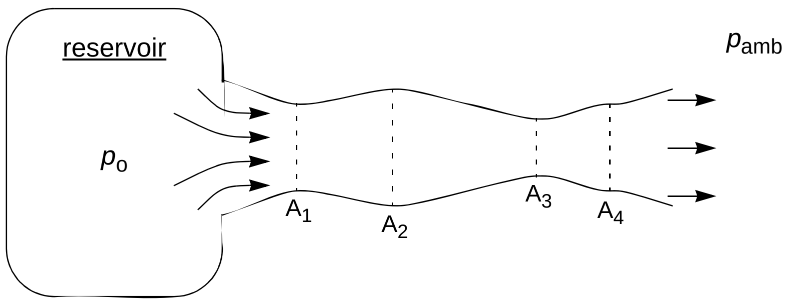
$$\left(\frac{\dot{m}}{A}\right)_{max} = \frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R}}\frac{p_o}{\sqrt{T_o}}\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Note that the area choking occurs at is the sonic area, A^*. For air, $\gamma = 1.4$ and $R = \frac{\Re}{MW} = \frac{8314}{28.8}$, so the maximum mass flow rate is:

$$\left(\frac{\dot{m}}{A}\right)_{max} = 0.0404\frac{p_o}{\sqrt{T_o}}$$

This equation is know as “Fliegner’s Formula” and was discovered in the 19th century by brute-force experiment, trial, and error, rather than by the theoretical considerations we have gone through.

Let us now consider an entire flow through a variable area duct fed from a constant stagnation condition reservoir:



If this flow is choked (due to p_{amb} being below some critical value), we know that it must occur where the flow is at $M = 1$. And from the relation $\frac{dV}{V} = \frac{1}{M^2 - 1}\frac{dA}{A}$, we know that $M = 1$ can only occur in a constant area section. This means, either a maximum in area (A_2), a minimum in area (A_1, A_3), or an inflection point (A_4). If the flow is choked at one of the large areas (A_1 or A_2, for example), then the smaller area A_3 cannot pass the mass flow. Thus, if choking occurs, it ***must*** occur at the global minimum in area: A_3.

So, statements we can make:

- $M = 1$ (sonic flow) can only occur at a minima in area.
- If choking occurs, it occurs with $M = 1$ at the minimum area of entire flow.
- For a given $\dot{m}$, p_o, T_o, this minimum area is called the *sonic area* or *critical area*: A^*. The flow can never pass through a smaller area.

The sonic area A^* is a useful reference state, like stagnation conditions p_o, T_o or sonic conditions p^*, T^*. Because it is a function of $\dot{m}$, p_o, and T_o, sonic area A^* is a constant through an isentropic flow.

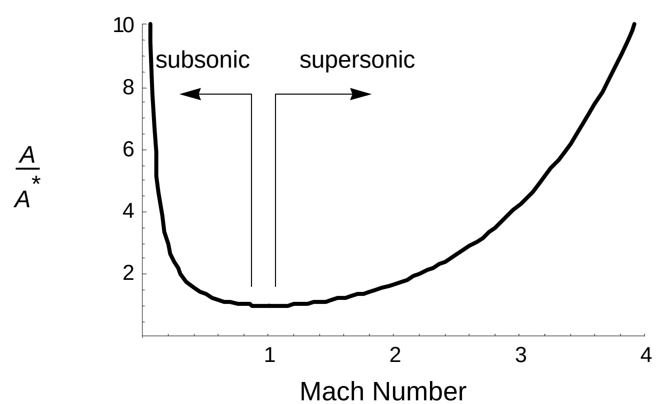
We can express the ratio of A to A^* as:

$$\frac{A}{A^*} = \frac{\left(\frac{\dot{m}}{A^*}\right)_{max}}{\frac{\dot{m}}{A}} = \frac{\sqrt{\frac{\gamma}{R}}\frac{p_o}{\sqrt{T_o}}\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}{\sqrt{\frac{\gamma}{R}}\frac{p_o}{\sqrt{T_o}}\frac{M}{\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}}$$

which simplifies to

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

This expression relates sonic area A^* to the local area A where the flow has Mach number M. We can plot $\frac{A}{A^*}$ as a function of Mach number



Note that $\frac{A}{A^*}$ is always greater than or equal to 1, since the area can never be smaller than the sonic area. Note also that, for a given area ratio $\frac{A}{A^*}$, there are ***two*** possible solutions: $M < 1$ and $M > 1$. The fact that there are subsonic and supersonic solutions for a given area ratio reflects the fact, as discussed earlier, that a sonic flow encountering a diverging area can go to either a subsonic or supersonic branch of the solution.

The fact that there are two possible solution branches for a given geometry of duct is a distinct feature of compressible fluid flow. Determining which branch of the solution is the correct one will be the subject of continued discussion throughout the remained of these notes.

5.5 Summary and Problem Solving

To summarize our results for isentropic flow: from the energy equation, we know T_o is a constant for adiabatic flow, and if the flow is isentropic as well, p_o and ρ_o are also constant. For fixed stagnation conditions, the sonic area or critical area A^* of the flow is also a constant. All of these constants are related to their local values by the local value of Mach number:

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}M^2$$

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma - 1}}$$

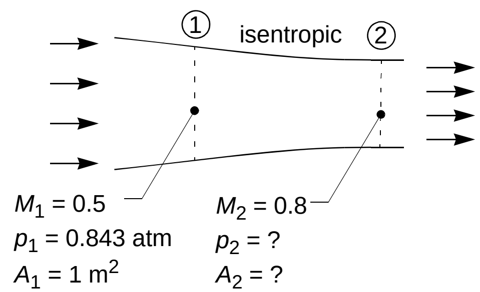
$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{\gamma - 1}{2}M^2\right)\right]^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

With these relations, we can solve for the conditions at any point in an isentropic flow if we know the conditions anywhere else in the same flow.

Our technique in problem solving will be to find the Mach number M and the reference conditions (stagnation or sonic properties and the critical area A^*) from the given information at a point in the flow. In order to find the properties at a new point in the flow, we need the Mach number at that new point. Once the Mach number at the new location is known, we can apply the relations above to find everything else.

5.5.1 Numerical Example: Solving an Isentropic Flow

Problem: Consider the following isentropic flow of air through a duct.



We are given the Mach number, pressure, and area at station 1 and asked to find the pressure and area at station 2 where the Mach number is now 0.8.

Solution:

Step 1: From the information given at station 1, find reference conditions.

To find stagnation pressure:

$$\frac{p_o}{p_1} = \left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{\frac{\gamma}{\gamma - 1}}$$

$$p_o = (0.843)\left(1 + \frac{1.4 - 1}{2}0.5^2\right)^{\frac{1.4}{1.4 - 1}} = 1.00 \text{ atm}$$

To find sonic area:

$$\frac{A_1}{A^*} = \frac{1}{M_1}\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{\gamma - 1}{2}M_1^2\right)\right]^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

$$A^* = \frac{A_1}{\frac{1}{M_1}\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{\gamma - 1}{2}M_1^2\right)\right]^{\frac{\gamma + 1}{2(\gamma - 1)}}}$$

$$A^* = \frac{1}{\frac{1}{0.5}\left[\left(\frac{2}{1.4 + 1}\right)\left(1 + \frac{\gamma - 1}{2}0.5^2\right)\right]^{\frac{1.4 + 1}{2(1.4 - 1)}}} = 0.746 \text{ m}^2$$

Thus, if the flow were stagnated, it would be at a pressure of 1 atm. We might speculate that this flow originated from a reservoir at 1 atm, but it does not really matter if this flow actually stagnates at some point or not. The p_o = 1 atm reference state applies to the entire flow. Further, if the flow were brought to sonic (M = 1), the area of the flow would be A^* = 0.746 m^2. Again, the flow may never reach sonic, but we will use this reference state.

Step 2: Using reference conditions (p_o, A^*) and the Mach number at station 2, find the conditions at station 2 (p_2, A_2).

To find pressure p_2:

$$\frac{p_o}{p_2} = \left(1 + \frac{\gamma - 1}{2}M_2^2\right)^{\frac{\gamma}{\gamma - 1}}$$

$$p_2 = \frac{p_o}{\left(1 + \frac{\gamma - 1}{2}M_2^2\right)^{\frac{\gamma}{\gamma - 1}}} = 0.656 \text{ atm}$$

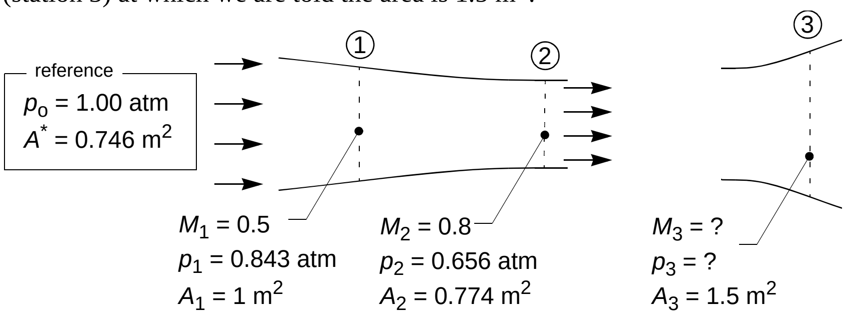
To find area A_2:

$$\frac{A_2}{A^*} = \frac{1}{M_2}\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{\gamma - 1}{2}M_2^2\right)\right]^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

$$A_2 = A^*\frac{1}{M_2}\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{\gamma - 1}{2}M_2^2\right)\right]^{\frac{\gamma + 1}{2(\gamma - 1)}} = 0.774 \text{ m}^2$$

We know that for a subsonic flow, if M increases, the area must decrease and pressure should also decrease. Indeed, this was the case in going from station 1 to station 2.

What if there is a third station along the isentropic flow through the duct (station 3) at which we are told the area is 1.5 m^2?



The same reference states still apply. Thus, knowing the area at station 3 (A_3) and the sonic area (A^*), we should be able to solve for Mach number:

$$\frac{A_3}{A^*} = \frac{1}{M_3}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_3^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Unfortunately, we cannot solve this equation explicitly for M_3. We can guess values of M_3 and iterate until the right hand side generates a value of $\frac{A_3}{A^*} = \frac{1.5}{0.746} = 2.009$.

Guess $M_3 = 0.2 \Rightarrow \frac{A_3}{A^*} = 2.963$

This value is too large, implying that the flow at station 3 has not decelerated as low as Mach 0.2. Thus, we will try a larger value of M_3 for our next guess:

Guess $M_3 = 0.3 \Rightarrow \frac{A_3}{A^*} = 2.035$

Closer, but still too large, so we will try $M_3 = 0.31 \Rightarrow \frac{A_3}{A^*} = 1.976$. Since these values bound the actual value of $\frac{A_3}{A^*} = 2.009$, we can interpolate between $M_3 = 0.30$ and $M_3 = 0.31$:

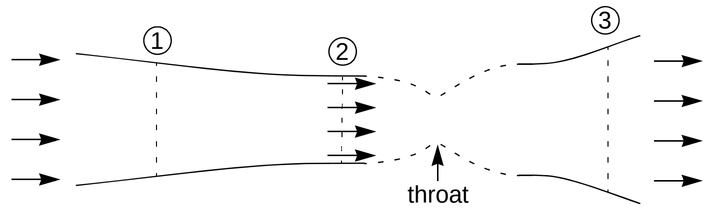
$$M_3 = 0.30 + \frac{2.009-2.035}{1.976-2.035}(0.31-0.30) = 0.304$$

Using this value of $M_3 = 0.304$, we can find the pressure at station 3:

$$p_3 = \frac{p_o}{\left(1+\frac{\gamma-1}{2}M_3^2\right)^{\frac{\gamma}{\gamma-1}}} = 0.938 \text{ atm}$$

So, we have solved for the conditions at station 3. Note that as the flow decelerates to a Mach number of about 0.3, it regains nearly its full stagnation pressure.

Are we missing something? Recall from Section 5.3, for a given area ratio $\frac{A}{A^*}$, there are ***two*** possible solutions: *subsonic* and *supersonic*. Here, we have found the subsonic solution. There also exists a supersonic solution corresponding to $\frac{A_3}{A^*} = 2.009$. Could the flow have reached supersonic speeds at station 3? As we have seen, in order for flow to transition from subsonic to supersonic, it must pass though a constant area section or throat where the flow is sonic. Once it has passed through this minimum, it can move to the supersonic branch of the solution in the diverging area section following the throat. Since we were not told the details of the flow between station 2 and 3, it is possible that a throat was encountered, permitting the flow to transition from subsonic to supersonic:



If such a throat existed, we would already know its area for this problem: $A_{\text{throat}} = A^* = 0.746 \text{ m}^2$.

To find the supersonic solution for station 3, we will make an initial guess of $M_3 = 3$:

$$\text{Guess } M_3 = 3 \Rightarrow \frac{A_3}{A^*} = 4.235$$

Since this area ratio is too large, we will guess a lower Mach number;

$$\text{Guess } M_3 = 2 \Rightarrow \frac{A_3}{A^*} = 1.688$$

Iterating toward the value $\frac{A_3}{A^*} = 2.009$:

$$\text{Guess } M_3 = 2.5 \Rightarrow \frac{A_3}{A^*} = 2.637$$

$$\text{Guess } M_3 = 2.25 \Rightarrow \frac{A_3}{A^*} = 2.096$$

$$\text{Guess } M_3 = 2.2 \Rightarrow \frac{A_3}{A^*} = 2.005$$

Interpolating between these last two values:

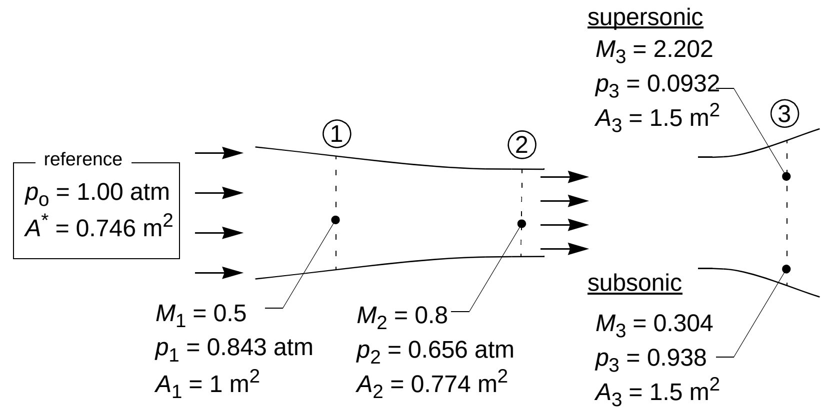
$$M_3 = 2.2 + \frac{2.009 - 2.005}{2.096 - 2.005}(2.25 - 2.20) = 2.202$$

Knowing the value of M_3, we can find p_3:

$$p_3 = \frac{p_o}{\left(1 + \frac{\gamma - 1}{2}M_3^2\right)^{\frac{\gamma}{\gamma - 1}}} = 0.0932 \text{ atm}$$

Note that the pressure for the supersonic solution is much lower; this large drop in pressure is necessary to accelerate the flow to supersonic speeds.

To summarize the results for this flow:



Note that we listed both the subsonic and supersonic solutions. Unless we are given additional information, we cannot say which solution will occur, so we must consider both of them as possibilities.

As interpolation of the $\frac{A}{A^*}$ relation becomes tedious, it is convenient to have a table of prepared values of $\frac{A}{A^*}$. Such a table, listing the isentropic ratios $\frac{p}{p_o}$, $\frac{\rho}{\rho_o}$, and $\frac{T}{T_o}$ as well as $\frac{A}{A^*}$ as functions of Mach number for $\gamma = 1.4$, is listed in Appendix Table A2.

5.6 Compressibility

As mentioned in the Introduction, compressible fluid flow is the study of flow in which the change in density, or *compressibility*, of a fluid cannot be neglected due to the large velocity of the flow. But when does the incompressible assumption used in low speed fluid dynamics break down, and how bad of an approximation is incompressible flow when we encounter high velocity?

To answer these questions, we will return to the differential form of the momentum equation:

$$\frac{dp}{\rho} + VdV = 0$$

If we assume that the density is constant (ρ = constant), then we can integrate the momentum equation from an initial state 1 to a final state 2:

$$\int_1^2 \frac{dp}{\rho} + \int_1^2 VdV = \int_1^2 0$$

$$\frac{1}{\rho}(p_2 - p_1) + \frac{1}{2}(V_2^2 - V_1^2) = 0$$

Or $$p_1 + \frac{1}{2}\rho_1 V_1^2 = p_2 + \frac{1}{2}\rho_2 V_2^2 = \text{constant}$$

Which is a form of Bernoulli's equation. Defining a stagnation condition p_o when $V = 0$:

$$p + \frac{1}{2}\rho V^2 = p_o$$

$$\frac{p_o - p}{\frac{1}{2}\rho V^2} = 1$$

This relation says that a change in pressure is directly converted into kinetic energy of the flow. Pressure is a *dynamic* variable, but not a *thermodynamic* variable in incompressible flow. Thus, we do not even need to invoke the energy equation in a compressible flow.

Of course, Bernoulli's equation is wrong and becomes increasingly inaccurate as flow velocity increases. But by how much?

To answer, let us compare the expression above to the expression for stagnation pressure for an isentropic, compressible flow:

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

To get a feel for how this relation holds for a low velocity (low Mach number), we can expand the right hand side using the binomial theorem:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

which converges for $x < 1$. In our expression for $\frac{p_o}{p}$, $x = \frac{\gamma - 1}{2}M^2$ and $n = \frac{\gamma}{\gamma - 1}$. So,

$$\frac{p_o}{p} = 1 + \frac{\gamma - 1}{2}\frac{\gamma}{\gamma - 1}M^2 + \frac{1}{2}\frac{\gamma}{\gamma - 1}\left(\frac{\gamma}{\gamma - 1} - 1\right)\left(\frac{\gamma - 1}{2}\right)^2 M^4 + O(M^6)$$

which simplifies to:

$$\frac{p_o}{p} = 1 + \frac{\gamma}{2}M^2 + \frac{\gamma}{8}M^4 + O(M^6)$$

Multiplying through by p:

$$p_o = p + \frac{\gamma}{2}pM^2 + \frac{\gamma}{8}pM^4 + O(M^6)$$

Using the relationship $\gamma p M^2 = \rho V^2$:

$$\frac{p_o - p}{\frac{1}{2}\rho V^2} = 1 + \frac{1}{4}M^2 + O(M^4)$$

So, as $M \to 0$, $\dfrac{p_o - p}{\frac{1}{2}\rho V^2} \to 1$, and we "recover" Bernoulli's equation. As Mach number increases, $\dfrac{p_o - p}{\frac{1}{2}\rho V^2} > 1$, because changes in pressure are not just accelerating the flow (i.e., increasing kinetic energy) but also to *expanding* the flow (i.e., decreasing density). To give a feel for when these effect are significant, here are some numbers: At Mach 0.3, $\dfrac{p_o - p}{\frac{1}{2}\rho V^2} \approx 1.022$, so compressibility only introduces a 2% error into the incompressible assumption! At Mach 1.0, $\dfrac{p_o - p}{\frac{1}{2}\rho V^2} \approx 1.25$, or a 25% error. Thus, at subsonic speeds, the incompressible flow assumption may not be too bad of an approximation if we do not require a very accurate answer. In low-speed aircraft design, for example, it is not uncommon to first design an airplane using incompressible flow assumptions, and then apply a correction factor to the design to account for the relatively minor effects of compressibility.

For supersonic flow, however, the compressibility of the fluid cannot be neglected and Bernoulli's equation is completely inappropriate.