Be sure this exam has 8 pages including the cover

The University of British Columbia

MATH 103

Midterm Exam I – Feb 8, 2012

Family Name ______________________ **Given Name** ______________________

Student Number ____________________ **Signature** ______________________

This exam consists of **5** questions. No notes nor calculators. Note the number of marks for each question. Use your time wisely.

Problem	max score	score
1.	15	
2.	8	
3.	10	
4.	12	
5.	5	
total	50	

(15 points) 1. Compute the following sums and integrals.

(a) $\sum_{k=1}^{7}(k^2-2k+2)$.

(b) $\sum_{k=2}^{7}2^k$.

(c) $\int_1^3 \frac{1}{x^3}dx$.

(d) $\int_0^1(\sin(3\pi x)+\cos x)dx$.

(e) $\int_{-1}^{x}(1+2t-3t^2)dt$.

Solution:

(a)

$$\begin{aligned}\sum_{k=1}^{7}(k^2-2k+2) &= \sum_{k=1}^{7}k^2 - 2\sum_{k=1}^{7}k + \sum_{k=1}^{7}2\\ &= \frac{7(7+1)(14+1)}{6} - 2\frac{7(7+1)}{2} + 14\\ &= 140-56+14 = 98.\end{aligned}$$

(b)

$$\sum_{k=2}^{7}2^k = \sum_{k=0}^{7}2^k - \sum_{k=0}^{1}2^k = \frac{1-2^{7+1}}{1-2} - (1+2) = 255-3 = 252.$$

(c)

$$\int_1^3 \frac{1}{x^3}dx = \int_1^3 x^{-3}dx = \left.\frac{x^{-2}}{-2}\right|_1^3 = \frac{3^{-2}}{-2} - \frac{1^{-2}}{-2} = -\frac{1}{18} + \frac{1}{2} = \frac{4}{9}.$$

(d)

$$\begin{aligned}\int_0^1(\sin(3\pi x)+\cos x)dx &= \left.\left(\frac{-\cos(3\pi x)}{3\pi} + \sin x\right)\right|_0^1\\ &= \frac{-\cos(3\pi)}{3\pi} + \sin 1 - \left(\frac{-\cos(0)}{3\pi} + \sin(0)\right)\\ &= \frac{1}{3\pi} + \sin 1 + \frac{1}{3\pi} - 0 = \frac{2}{3\pi} + \sin 1.\end{aligned}$$

(e)

$$\begin{aligned}\int_{-1}^{x}(1+2t-3t^2)dt &= \left.\left(t+t^2-t^3\right)\right|_{-1}^{x}\\ &= x+x^2-x^3-\left(-1+(-1)^2-(-1)^3\right)\\ &= x+x^2-x^3-(-1+1+1)\\ &= x+x^2-x^3-1\end{aligned}$$

(8 points) 2. The speed of a car (km/h) is given by the expression

$$v(t) = t^3 + 2t + 10, \qquad 0 < t < 2,$$

where t is time in hours. Use this expression to find

(a) The average velocity of the car during this time period.

(b) The acceleration of the car over this time period.

(c) The average acceleration of the car over this time period.

(d) The total displacement over this time period.

Solution:

(a)

$$\begin{aligned}\bar{v} &= \frac{1}{2-0}\int_0^2 v(t)dt \\ &= \frac{1}{2}\int_0^2 \left(t^3 + 2t + 10\right) dt \\ &= \frac{1}{2}\left(\frac{1}{4}t^4 + t^2 + 10t\right)\Big|_0^2 \\ &= \frac{1}{2}\left(\frac{1}{4}16 + 4 + 20 - 0\right) \\ &= 14 \text{ km/h}\end{aligned}$$

(b)

$$a(t) = \frac{dv}{dt} = \left(3t^2 + 2\right) \text{ km/h}^2$$

(c)

$$\begin{aligned}\bar{a} &= \frac{1}{2-0}\int_0^2 a(t)dt \\ &= \frac{1}{2}\int_0^2 \frac{dv}{dt}dt \\ &= \frac{1}{2}\, v(t)|_0^2 \\ &= \frac{1}{2}\left(t^3 + 2t + 10\right)\Big|_0^2 \\ &= \frac{1}{2}\Big((8 + 4 + 10) - (10)\Big) \\ &= 6 \text{ km/h}^2\end{aligned}$$

(d)

$$\Delta x = x(2) - x(0) = \int_0^2 \frac{dx}{dt}dt = \int_0^2 v(t)dt = 2\bar{v} = 28 \text{ km}$$

(10 points) 3. Compute the area between the graphs of the two functions $f(x) = \frac{1}{2}x^2 - 2$ and $g(x) = -\frac{1}{2}x^2 + x$ by completing the following steps.

(a) Find the intersection points of $f(x)$ and $g(x)$. Draw a graph of $f(x)$ and $g(x)$ with intersection points marked.

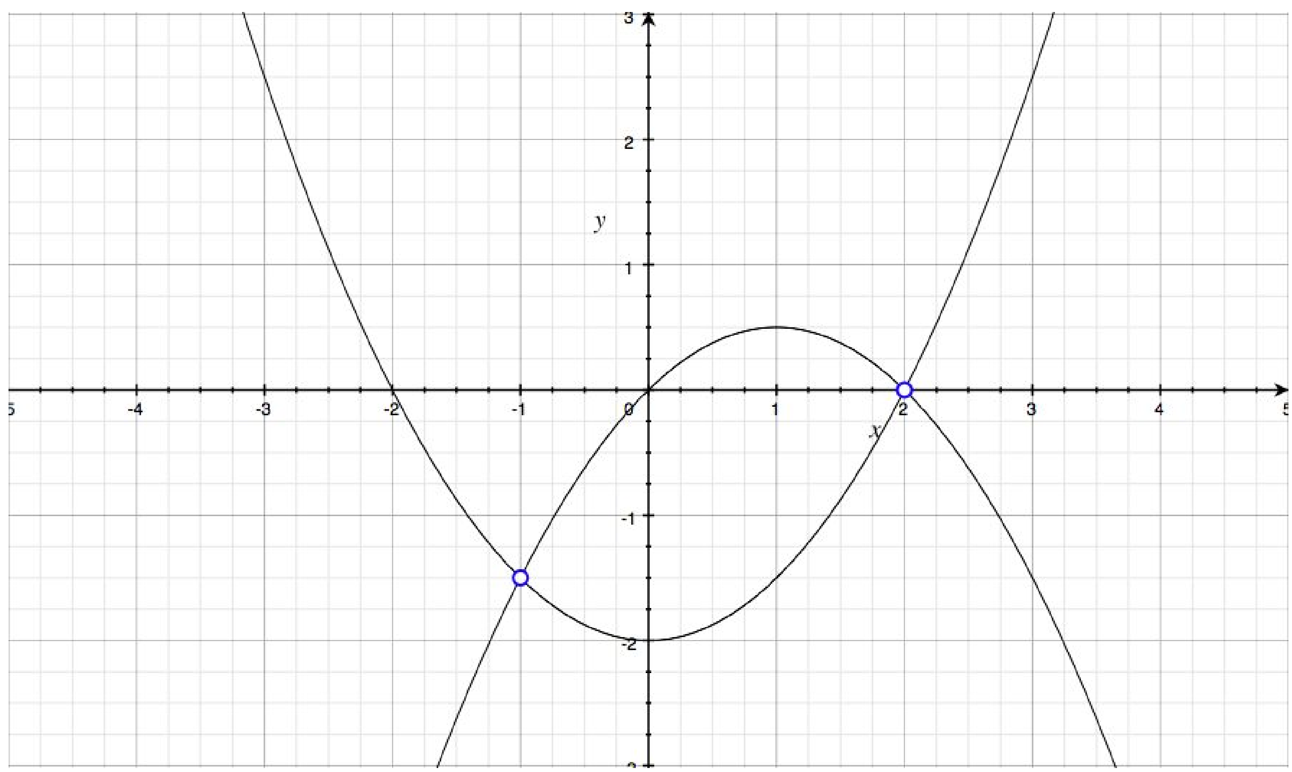
(b) Find the area by first writing down the integral. Compute the integral by using fundamental theorem of calculus.

Solution:

(a) The intersection points of $f(x)$ and $g(x)$ are solutions to the equation $f(x) = g(x)$.

$$\begin{aligned} \frac{1}{2}x^2 - 2 &= -\frac{1}{2}x^2 + x \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2, -1 \end{aligned}$$

The two intersection points are $(-1, -\frac{3}{2})$, $(2, 0)$.



(b) Let A denote the area between $f(x)$ and $g(x)$. From the graph, the graph of $g(x)$

is above the one of $f(x)$ in the interval $(-1, 2)$.

$$\begin{aligned} A &= \int_{-1}^{2} [g(x) - f(x)]dx \\ &= \int_{-1}^{2} \left[\left(-\frac{1}{2}x^2 + x\right) - \left(\frac{1}{2}x^2 - 2\right)\right] dx \\ &= \int_{-1}^{2} (-x^2 + x + 2)dx \\ &= -\frac{x^3}{3} + \frac{x^2}{2} + 2x\Big|_{-1}^{2} \\ &= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) \\ &= \frac{9}{2} \end{aligned}$$

The area between $f(x)$ and $g(x)$ is $9/2$.

(12 points) 4. Consider the area enclosed by the graph of the function $f(x) = 1 - e^{-2x}$, the x-axis, $x = 0$ and $x = 1$. Answer the following questions

(a) Use fundamental theorem of calculus to compute this area.

(b) Next divide the x-axis between $x = 0$ and $x = 1$ into N intervals of equal length Δx. What is the value of Δx?

(c) Let the k-th interval be between x_k and x_{k+1} ($x_0 = 0$, $x_N = 1$). What is x_k in terms of k and N?

(d) Let the k-th rectangle be of height $f(x_k)$ and width Δx. Find the area of k-th rectangle in terms of k and N.

(e) Write the approximate area S_N using the Σ symbol. Compute your sum to find an expression for S_N in terms of N.

(f) Show that as $N \to \infty$, S_N approaches the value you computed in using the fundamental theorem of calculus in Part (a). (Hint: for x small, you can use the approximation $e^{-x} \approx 1 - x$).

Solution:

a)

$$\int_0^1 (1 - \exp(-2x))\, dx = x + \left.\frac{\exp(-2x)}{2}\right|_0^1 = \frac{1}{2}(1 + \exp(-2))$$

b) $\Delta x = \frac{1}{N}$

c) k must index from 0 to $N - 1$ based on the instructions. In this case $x_k = k\Delta x = k/N$.

d) Since $f(x_k) = 1 - \exp(-2k/N)$, the area

$$\Delta A_k = (1 - \exp(-2k/N))\frac{1}{N}$$

e)

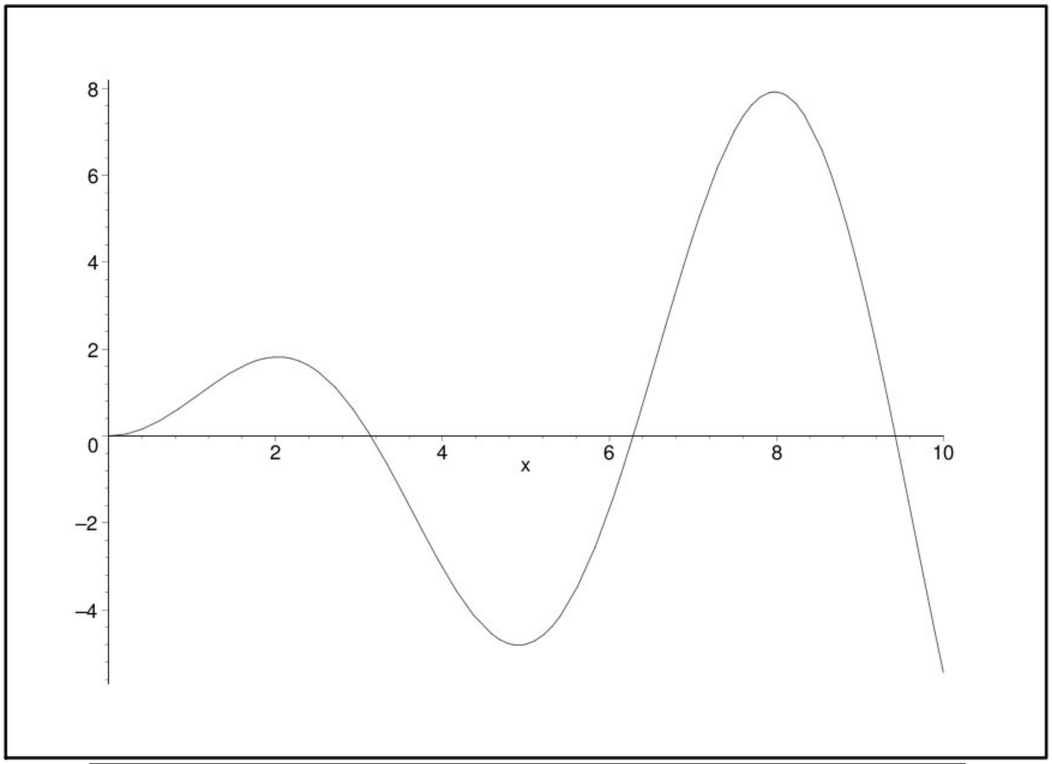
$$S_N = \sum_{k=0}^{N-1} \Delta A_k = \sum_{k=0}^{N-1} \left(1 - \exp\left(\frac{-2k}{N}\right)\right)\frac{1}{N} = 1 - \frac{1}{N}\sum_{k=0}^{N-1}\left[\exp\left(\frac{-2}{N}\right)\right]^k$$

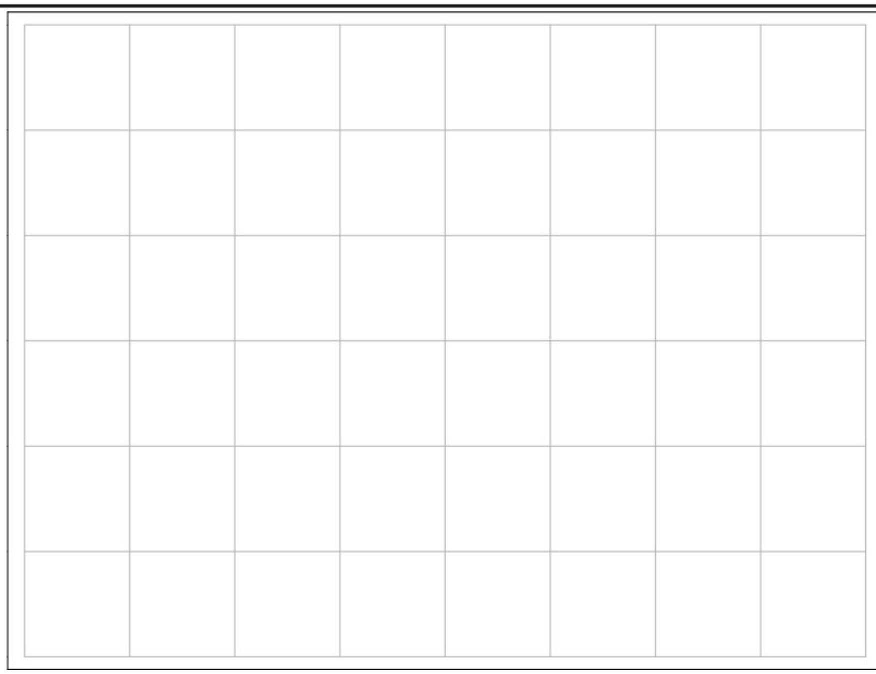
$$= 1 - \frac{1}{N}\frac{1 - \left[\exp\left(\frac{-2}{N}\right)\right]^N}{1 - \exp\left(\frac{-2}{N}\right)} = 1 - \frac{1}{N}\frac{1 - \exp(-2)}{1 - \exp\left(\frac{-2}{N}\right)}$$

f) Using the fact that $1 - \exp(-2/N) \approx 2/N$, the limit can be evaluated as

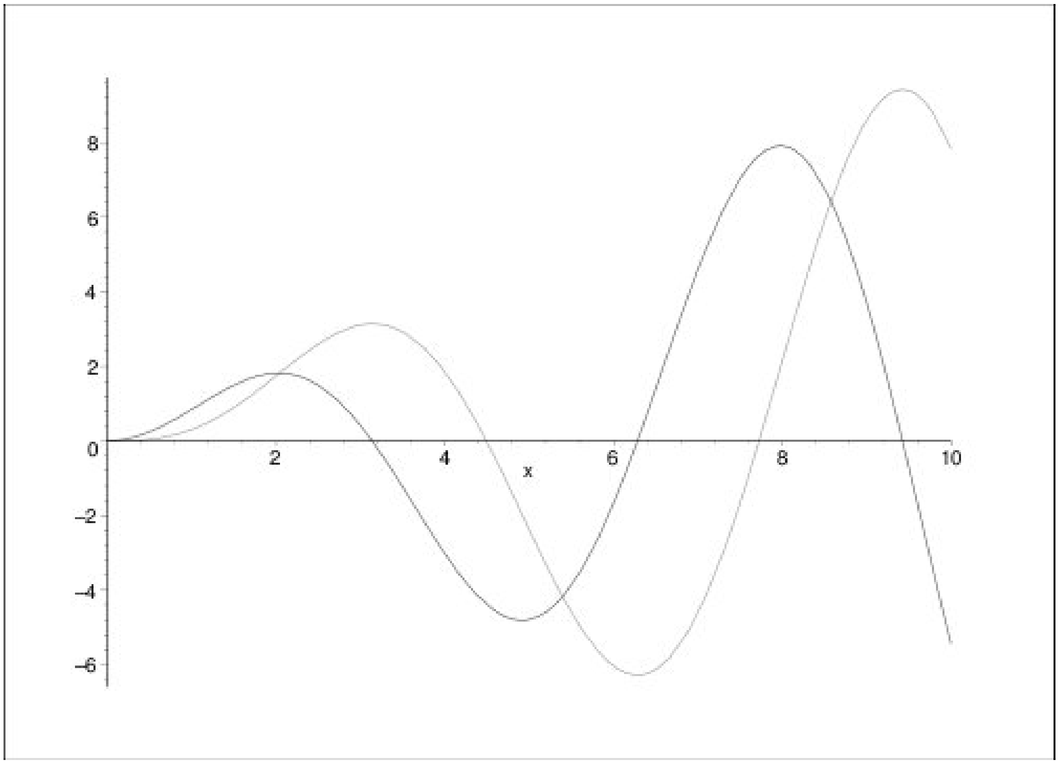
$$\lim_{N\to\infty} S_N = 1 - \frac{1}{2}(1 - \exp(-2)) = \frac{1}{2}(1 + \exp(-2))$$

(5 points) 5. Consider the function shown in the box. Use the sketch of this function $y = f(x)$ to draw a sketch of the graph of the related function $F(x) = \int_0^x f(t)dt$ on the grid provided. (Note that $F(0) = 0$.)





Solution:


8
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6
8
10
x