(B)

Question 1. (4 points) A moving object has position vector

$$\vec{r}(t) = (2t^3 - 6t - 3)\,\hat{\imath} + (3t^2 - 6t + 4)\,\hat{\jmath} + (9t^2 - 18t + 2)\,\hat{k}\,.$$

(a) Find the velocity and acceleration vectors.

(b) Are there any times at which the object is at rest? If yes, what is the acceleration at those times?

a) $\boxed{\vec{v}(t) = \vec{r}\,'(t) = (6t^2 - 6)\hat{\imath} + (6t - 6)\hat{\jmath} + (18t - 18)\hat{k}}$

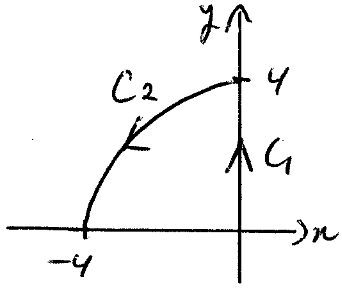
$\boxed{\vec{a}(t) = \vec{v}\,'(t) = 12t\,\hat{\imath} + 6\hat{\jmath} + 18\hat{k}}$

b) $\vec{v}(t) = \vec{0}$ if $\boxed{t = 1}$

$\boxed{\vec{a}(1) = 12\hat{\imath} + 6\hat{\jmath} + 18\hat{k}}$

(B)

Question 2. (5 points) Calculate the line integral of $\vec{F}(x,y) = 4x\,\hat{\imath} + (2-2y)\,\hat{\jmath}$ along the path C, where C consists of the straight line from $(0,0)$ to $(0,4)$ and then a circular arc (part of the circle $x^2+y^2=16$) directly from $(0,4)$ to $(-4,0)$.



C_1 $\quad x=0,\ y=t \quad 0 \le t \le 4$

$\vec{r}(t) = t\hat{\jmath}, \quad \vec{r}\,'(t) = \hat{\jmath}$

$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^4 (2-2t)\hat{\jmath}\cdot\hat{\jmath}\,dt = \int_0^4 (2-2t)\,dt$

$= 2t - t^2 \Big|_0^4 = \boxed{-8}$

C_2 $\quad x = -4\sin t,\ y = 4\cos t, \quad 0 \le t \le \pi/2$

$\vec{r}(t) = -4\sin t\,\hat{\imath} + 4\cos t\,\hat{\jmath}, \quad \vec{r}\,'(t) = -4\cos t\,\hat{\imath} - 4\sin t\,\hat{\jmath}$

$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} (-16\sin t\,\hat{\imath} + (2-8\cos t)\hat{\jmath})\cdot(-4\cos t\,\hat{\imath} - 4\sin t\,\hat{\jmath})\,dt$

$= \int_0^{\pi/2} (96\sin t\cos t - 8\sin t)\,dt$

$= 48\sin^2 t + 8\cos t \Big|_0^{\pi/2} = (48+0) - (0+8) = \boxed{40}$

$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = 40 - 8 = \boxed{32}$

OR easier way: $\quad \vec{F} = 4x\hat{\imath} + (2-2y)\hat{\jmath} = \nabla f$ where $f(x,y) = 2x^2 + 2y - y^2$, so $\vec{F}$ is path-independent and we can take any path from $(0,0)$ to $(-4,0)$, like the straight line $\vec{r}(t) = -t\hat{\imath} \quad 0 \le t \le 4, \quad \vec{r}\,'(t) = -\hat{\imath}$

so $\int_C \vec{F}\cdot d\vec{r} = \int_0^4 (-4t\hat{\imath})\cdot(-\hat{\imath})\,dt = \int_0^4 4t\,dt = 2t^2\Big|_0^4 = \boxed{32}$

OR easiest way: $\quad \vec{F} = \nabla f$ where $f(x,y) = 2x^2 + 2y - y^2$

then $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(-4,0) - f(0,0)$

$= 2(-4)^2 + 0 - 0 = \boxed{32}$

Question 3. (6 points) Exactly one of the following vector fields is a gradient field. Determine which one it is and explain why the others are not gradient vector fields. What is the potential function for the gradient vector field ?

$\vec{F}(x,y) = xy^3\,\hat{\imath} + x^2\,\hat{\jmath}$

$\vec{G}(x,y) = (2x - \cos(y))\,\hat{\imath} + x\sin(y)\,\hat{\jmath}$

$\vec{H}(x,y) = \cos(y)\,\hat{\imath} + \sin(x)\,\hat{\jmath}$

the vector fields are defined everywhere in $\mathbb{R}^2$, so we can check if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ or not

$\vec{F} = xy^3\,\hat{\imath} + x^2\,\hat{\jmath} \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 3xy^2 \neq 0$

not gradient field

$\vec{G} = (2x - \cos y)\,\hat{\imath} + x\sin y\,\hat{\jmath} \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin y - \sin y = 0$

so $\boxed{\vec{G} \text{ is gradient field}}$

$\vec{H} = \cos y\,\hat{\imath} + \sin x\,\hat{\jmath} \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos x + \sin y \neq 0$

not gradient field

$\vec{G} = \nabla g$ so $g(x,y) = \int (2x - \cos y)\,dx + f(y)$

$= x^2 - x\cos y + f(y)$

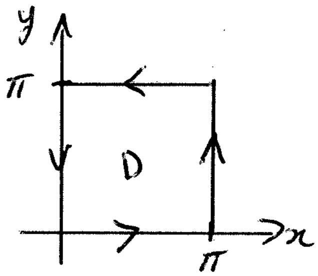
then $\frac{\partial g}{\partial y} = x\sin y + f'(y) = Q = x\sin y \Rightarrow f'(y) = 0$

thus $f(y) = C$

so $\boxed{g(x,y) = x^2 - x\cos y + C}$ is the potential function

B

Question 4. (5 points) Let C be the square with corners at $(0,0)$, $(\pi,0)$, (π,π), and $(0,\pi)$, oriented counter-clockwise. If $\vec{F}(x,y) = x\cos(y)\,\hat{\imath} + 2y\sin(x)\,\hat{\jmath}$, calculate $\oint_C \vec{F}\cdot d\vec{r}$ using Green's Theorem.



Green's Theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$P(x,y) = x\cos y$

$Q(x,y) = 2y\sin x$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{\pi}\int_0^{\pi} (2y\cos x + x\sin y)\,dx\,dy$$

$$= \int_0^{\pi} \left(2y\sin x + \tfrac{1}{2}x^2\sin y\Big|_0^{\pi}\right) dy$$

$$= \int_0^{\pi} \tfrac{1}{2}\pi^2 \sin y\,dy$$

$$= \tfrac{1}{2}\pi^2\left(-\cos y\Big|_0^{\pi}\right)$$

$$= \boxed{\pi^2}$$