104:2006

1a)

$$f(-2)=\lim_{x\to -2} f(x)=\lim_{x\to -2^-} f(x)=\lim_{x\to -2^+} f(x)$$

$$c=\lim_{x\to -2}\frac{x^2-3x-10}{x+2}=\lim_{x\to -2}\frac{\cancel{(x+2)}(x-5)}{\cancel{x+2}}=\lim_{x\to -2} x-5$$

$$c=-7$$

1b)

$$\lim_{t\to 0}\frac{\sqrt{t+9}-3}{\sqrt{t}}=\lim_{t\to 0}\frac{\sqrt{t+9}-\sqrt{9}}{\sqrt{t}}=\lim_{t\to 0}\sqrt{1+9/t}-\sqrt{9/t}=$$

$$\lim_{t\to 0}\left(\sqrt{1+9/t}-\sqrt{9/t}\right)\frac{\sqrt{1+9/t}+\sqrt{9/t}}{\sqrt{1+9/t}+\sqrt{9/t}}=\lim_{t\to 0}\frac{(1+\cancel{9/t})-(\cancel{9/t})}{\sqrt{1+9/t}+\sqrt{9/t}}=$$

$$\lim_{t\to 0}\frac{1}{\sqrt{1+9/t}+\sqrt{9/t}}=\frac{1}{\infty+\infty}=0$$

1c)

$$\lim_{h\to 0}\frac{(h+3)^2-9}{(h-5)^2-25}=\lim_{h\to 0}\frac{h^2+6h+\cancel{9-9}}{h^2-10h+\cancel{25-25}}=$$

$$\lim_{h\to 0}\frac{\cancel{h}(h+6)}{\cancel{h}(h-10)}=\lim_{h\to 0}\frac{h+6}{h-10}=\frac{6}{-10}=-\frac{3}{5}$$

1d)

$$A=A_0e^{rt}$$

$$A=3A_0 \text{ at } t=t^*$$

$$3\cancel{A_0}=\cancel{A_0}e^{rt^*}$$

$$\ln 3=rt^*$$

$$t^*=\frac{\ln 3}{r}=\frac{\ln 3}{0.06}\approx 18.31 \text{ years}$$

1e)

$y - 3x = 1$ has slope $m = 3$

$$y = \frac{1}{4}(2x+1)^2$$

$$y' = \frac{1}{4}(2)(2)(2x+1) = 2x+1$$

$$y'(x_0) = m = 3$$

$$2x_0 + 1 = 3$$

$$x_0 = 1$$

$$y_0 = \frac{1}{4}(2x_0+1)^2 = \frac{9}{4}$$

1f)

$$y = e^{3x} + e^{-2x}$$

$$y' = 3e^{3x} - 2e^{-2x}$$

$$0 = 3e^{3x} - 2e^{-2x}$$

$$3e^{3x} = 2e^{-2x}$$

$$e^{5x} = \frac{2}{3}$$

$$5x = \ln\frac{2}{3} = \ln 2 - \ln 3$$

$$x = \frac{\ln 2 - \ln 3}{5} \approx -0.08109$$

1g)

$$f(x) = (\arcsin(x))^{-2}$$

$$f'(x) = -2(\arcsin(x))'(\arcsin(x))^{-3} = -2\frac{1}{\sqrt{1-x^2}}(\arcsin(x))^{-3}$$

$$f'(x) = \frac{-2}{\sqrt{1-x^2}(\arcsin(x))^3}$$

$$f'\left(\frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{1-1/2}\left(\arcsin\left(1/\sqrt{2}\right)\right)^3} = \frac{-2\sqrt{2}}{(\pi/4)^3} = \frac{-128\sqrt{2}}{\pi^3}$$

1h)

$g(x) = \sqrt{1+3f(x)} = (1+3f(x))^{1/2}$

$g'(x) = \frac{1}{2}(3f'(x))(1+3f(x))^{-1/2}$

$g'(0) = \frac{1}{2}(3f'(0))(1+3f(0))^{-1/2}$

$g'(0) = \frac{1}{2}(12)(4)^{-1/2} = 3$

$m = 3$

$y = 3x + b$

$y(0) = g(0) = \sqrt{1+3f(0)} = 2$

$2 = 3(0) + b$

$b = 2$

$y = 3x + 2$

1i)

$y + \ln(y+3) = x^2$

$$\frac{dy}{dx} + \frac{dy}{dx}\frac{1}{y+3} = 2x$$

$$\frac{dy}{dx} = 2x\left(1 + \frac{1}{y+3}\right)^{-1} = 2x\left(\frac{y+3}{y+3} + \frac{1}{y+3}\right)^{-1}$$

$$\frac{dy}{dx} = 2x\frac{y+3}{y+4}$$

1j)

$$f(x) = \frac{1+\ln(x+1)}{x+1},\ x > -1$$

$$f'(x) = \frac{(x+1)^{-1}(x+1) - (1+\ln(x+1))}{(x+1)^2} = \frac{-\ln(x+1)}{(x+1)^2}$$

$f'(x) > 0$ when $\ln(x+1) < 0$

$-1 < x < 0$

1k)

$$f(x)=\frac{x}{x+2},\ x\neq -2$$

$$f'(x)=\frac{(x+2)-x}{(x+2)^2}=\frac{2}{(x+2)^2}=2(x+2)^{-2}$$

$$f''(x)=-4(x+2)^{-3}$$

$$f''(x)<0 \text{ when } (x+2)^{-3}>0$$

$$x+2>0$$

$$x>-2$$

1l)

$$f(x)=\frac{x}{x^2+4} \text{ on } [-1,5]$$

$$f'(x)=\frac{(x^2+4)-x(2x)}{(x^2+4)^2}=\frac{-x^2+4}{(x^2+4)^2}$$

$$f'(x)=0 \text{ when } -x^2+4=0$$

$x=\pm 2$, but $x=-2$ lies outside $[-1,5]$

$$f(-1)=-\frac{1}{5}, f(2)=\frac{1}{4}, f(5)=\frac{5}{29}$$

So $f(-1)=-\frac{1}{5}$ is the minimum on $[-1,5]$.

1m)

$$\sum_{n=0}^{\infty} ar^n=\frac{a}{1-r}$$

$$a=-\frac{4}{5}, r=-\frac{1}{5}$$

$$\sum_{n=0}^{\infty} -\frac{4}{5}\left(-\frac{1}{5}\right)^n=\frac{-4/5}{1-(-1/5)}=\frac{-4/5}{6/5}=-\frac{2}{3}$$

1n)

$$f(x)=(x+1)e^{-x}$$

$$c_2=\frac{f''(a)}{2!}=\frac{f''(0)}{2}$$

$$f'(x)=e^{-x}-(x+1)e^{-x}=-xe^{-x}$$

$$f''(x)=-e^{-x}+xe^{-x}$$

$$c_2=\frac{-e^0+0e^0}{2}=-\frac{1}{2}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{x+h}{1-3(x+h)} - \frac{x}{1-3x}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{x+h}{1-3x-3h} - \frac{x}{1-3x}\right) =$$

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(1-3x)}{(1-3x)(1-3x-3h)} - \frac{x(1-3x-3h)}{(1-3x)(1-3x-3h)}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{(-3x^2+x-3hx+h)-(-3x^2+x-3hx)}{(1-3x)(1-3x-3h)}\right) =$$

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{h}{(1-3x)(1-3x-3h)}\right) = \lim_{h\to 0} \frac{h}{h}\left(\frac{1}{(1-3x)(1-3x-3h)}\right) = \lim_{h\to 0} \frac{1}{(1-3x)(1-3x-3h)} = \frac{1}{(1-3x)^2}$$

3)

$q(p) = Ap + B,\ q(7) = 60,\ q(5) = 66,\ C(q) = 3q$

$66 - 60 = q(5) - q(7) = (5A + B) - (7A + B) = -2A$

$A = -3$

$60 = q(7) = -3(7) + B$

$B = 81$

$q(p) = -3p + 81$

Profit = Revenue − Cost

$P = pq - C(q) = pq - 3q = p(-3p+81) - 3(-3p+81) = -3p^2 + 90p - 243$

$P' = -6p + 90$

$P' = 0$

$6p = 90$

$p = 15$

4a)

$y = f(x) = \frac{2x}{x^2+1}$

$f'(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$

$f'(x) > 0$ when $-2x^2 + 2 > 0$

$2x^2 < 2$

$x^2 < 1$

$-1 < x < 1$

4b)

$$f(x)=\frac{x}{x+2},\ x\neq -2$$

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$$f''(x)=-e^{-x}+xe^{-x}$$

$$c_2=\frac{-e^0+0e^0}{2}=-\frac{1}{2}$$

$f'(x) = (-2x^2 + 2)(x^2 + 1)^{-2}$

$f''(x) = -4x(x^2 + 1)^{-2} + (-2x^2 + 2)(-2)(2x)(x^2 + 1)^{-3} = (x^2 + 1)^{-3}(-4x(x^2 + 1) - 4x(-2x^2 + 2))$

$f''(x) = (x^2 + 1)^{-3}(4x^3 - 12x) = \dfrac{4x^3 - 12x}{(x^2 + 1)^3} = \dfrac{4x(x^2 - 3)}{(x^2 + 1)^3}$

$f''(x) = 0$ when $4x = 0$, or $x^2 - 3 = 0$

Inflection points at $x = 0$, $x = -\sqrt{3}$ and $x = \sqrt{3}$

$-2 < -\sqrt{3}; f''(-2) = \dfrac{-8(4-3)}{75} = -\dfrac{8}{75} < 0$; f is concave down on $\left(-\infty, -\sqrt{3}\right)$

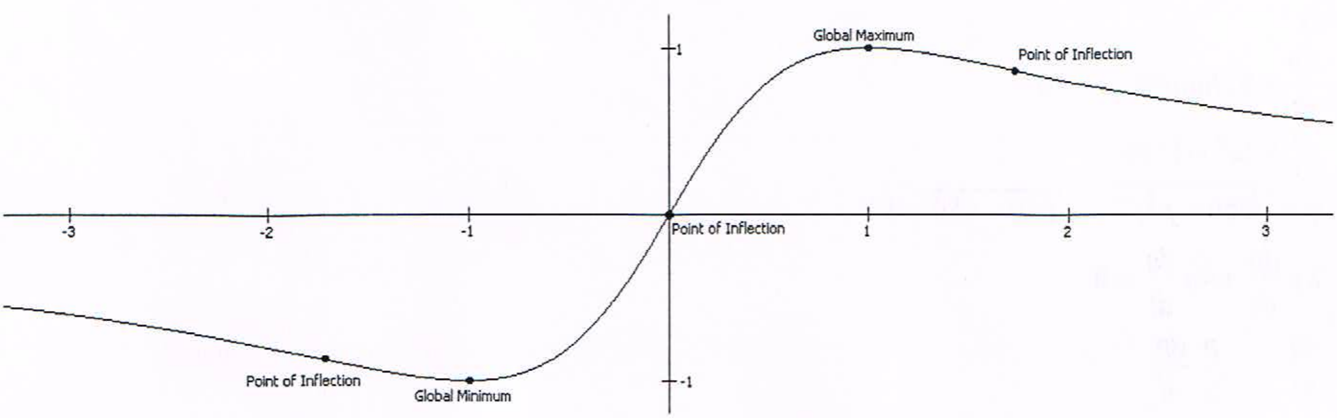
$-\sqrt{3} < -1 < 0; f''(-1) = \dfrac{-4(1-3)}{8} = 1 > 0$; f is concave up on $\left(-\sqrt{3}, 0\right)$

$0 < 1 < \sqrt{3}; f''(1) = \dfrac{4(1-3)}{8} = -1 < 0$; f is concave down on $\left(0, \sqrt{3}\right)$

$2 > \sqrt{3}; f''(2) = \dfrac{8(4-3)}{75} = \dfrac{8}{75} > 0$; f is concave up on $\left(\sqrt{3}, \infty\right)$

f is concave up on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$.

4c)



5)

$A = A_{base} + 4A_{side}$

$C = 5A_{base} + 2(4A_{side}) = 5A_{base} + 8A_{side}$

Let s be the width of the base of the box , and h be the height of the box.

$C = 5s^2 + 8sh$

$5s^2 + 8sh = 60$

$$h = \frac{60 - 5s^2}{8s}$$

$$V = lwh = s^2h = \frac{1}{8}(-5s^3 + 60s)$$

$$V' = \frac{1}{8}(-15s^2 + 60)$$

$V' = 0$

$15s^2 = 60$

$s = \pm 2$, but $s < 0$ doesn't make physical sense, so

$s = 2$ meters

$$h = \frac{60 - 5(4)}{16} = \frac{5}{2} = 2.5 \text{ meters}$$

2m $\times$2m $\times$2.5m are the dimensions of the box.

6)

$$\frac{dp}{dt} = \$2/\text{month},\ p = 30$$

$p^2 + 2q^2 = 1100$

$q = \sqrt{550 - p^2/2} = \sqrt{550 - 450} = 10$

$$2p\frac{dp}{dt} + 4q\frac{dq}{dt} = 0$$

$$\frac{dq}{dt} = -\frac{p}{2q}\frac{dp}{dt}$$

$R = pq$

$$R' = q\frac{dp}{dt} + p\frac{dq}{dt} = q\frac{dp}{dt} - \frac{p^2}{2q}\frac{dp}{dt} = 10(2) - \frac{900}{20}(2) = -\$70/\text{month}$$