CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP232 MATHEMATICS FOR COMPUTER SCIENCE

ASSIGNMENT 2 SOLUTIONS FALL 2012

1. In Problem 1(c) of Assignment 1 it was verified by truth table that

$$(p \wedge (\neg q \rightarrow \neg p)) \ \Rightarrow q \qquad (\star)$$

i.e., that $(p \wedge (\neg q \rightarrow \neg p)) \ \rightarrow q$ is a tautology.

Let LHS denote the left-hand side $(p \wedge (\neg q \rightarrow \neg p))$, and RHS the right-hand side q. State which of the following four approaches constitute a correct approach to prove $(\star)$, and which ones constitute a wrong approach:

(a) LHS $\Rightarrow$ RHS (b) $\neg$ LHS $\Rightarrow$ $\neg$ RHS (c) RHS $\Rightarrow$ LHS (d) $\neg$ RHS $\Rightarrow$ $\neg$ LHS

SOLUTION: The only correct approaches are (a) and (d).

Use a *direct proof* of one of the correct approaches to actually prove $(\star)$.

PROOF:

Approach (a) is possibly the easiest approach here: Assume the LHS is True. Then $p = T$ and $(\neg q \rightarrow \neg p) = T$. Now $(\neg q \rightarrow \neg p) \equiv (p \rightarrow q)$, and since $p = T$ it follows that $q = T$, *i.e.*, the RHS is True. QED

2. In Problem 1(a) of Assignment 1 it was verified by truth table that

$$((p \vee r) \wedge (q \vee r)) \equiv (p \wedge q) \vee r \qquad (\star\star)$$

i.e., $((p \vee r) \wedge (q \vee r)) \leftrightarrow (p \wedge q) \vee r$ is a tautology.

Let LHS denote the left-hand side $(p \vee r) \wedge (q \vee r)$, and RHS the right-hand side $(p \wedge q) \vee r$. State which of the following four approaches constitute a correct approach to prove $(\star\star)$, and which ones constitute a wrong approach:

(a) LHS $\Rightarrow$ RHS and RHS $\Rightarrow$ LHS (b) $\neg$ LHS $\Rightarrow$ $\neg$ RHS and $\neg$ RHS $\Rightarrow$ $\neg$ LHS

(c) LHS $\Rightarrow$ RHS and $\neg$ RHS $\Rightarrow$ $\neg$ LHS (d) RHS $\Rightarrow$ LHS and $\neg$ LHS $\Rightarrow$ $\neg$ RHS

SOLUTION: The only correct approaches are (a) and (b).

Use *direct proofs* in one of the correct approaches to actually prove $(\star\star)$. ("Cases" may be needed within the proofs.)

SOLUTION: Approach (a) is possibly the easiest approach here. This proof consists of two parts:

(i) Assume the LHS $(p \vee r) \wedge (q \vee r)$ is True. Then both $p \vee r$ and $q \vee r$ are True. This gives two cases to be considered: (i) r is True and (ii) both p and q are True. In either case the RHS $(p \wedge q) \vee r$ is True.

(ii) Assume the RHS $(p \wedge q) \vee r$ is True. This leads to two cases: both p and q are True or r is True, In either case the LHS $(p \vee r) \wedge (q \vee r)$ is True.

3. (a) Give a proof by *cases* to show that the equation

$$5x^2 \ + \ 4y^3 \ = \ 51$$

does not have solutions $x, y \in \mathbb{Z}^+$.

SOLUTION: It is clear that we only need to consider $x \leq 3$ and $y \leq 2$. This leads to six cases: $(x, y) = (1,1),\ (1,2),\ (2,1),\ (2,2),\ (3,1),\ (3,2)$. The cases $(x, y) = (2,1),\ (2,2)$ can be excluded immediately since they would give an even sum. Thus we need only consider $(x, y) = (1,1),\ (1,2),\ (3,1),\ (3,2)$. However, these give the sums 9, 37, 49, 77, respectively, that is, not 51.

(b) For $n \in \mathbb{Z}^+$ prove by contrapositive that if $2n^3 + 3n^2 + 4n + 5$ is odd then n is even.

SOLUTION: The contrapositive is: if n is odd then $2n^3 + 3n^2 + 4n + 5$ is even. So let n be odd. Then $2n^3$ and $4n$ are even, while both $3n^2$ and 5 are odd, so the overall sum is even.

4. For each of the statements below state whether it is *True* or *False*. If *True* then give a proof. If *False* then give a counterexample.

(a) The product of any two odd integers is odd.

SOLUTION: True: The two odd integers can be written as $n_1 = 2k_1 + 1$ and $n_2 = 2k_2 + 1$, so that their product equals $(2k_1 + 1)(2k_2 + 1) = 2(2k_1k_2 + k_1 + k_2) + 1$, which is odd.

(b) The sum of any even and any odd integer is odd.

SOLUTION: True: The even integer can be written as $n_1 = 2k_1$ and the odd integer as $n_2 = 2k_2 + 1$. Then $n_1 + n_2 = 2(k_1 + k_2) + 1$, which is odd.

(c) The difference of any two odd integers is odd.

False: Counterexample: Let $n_1 = 7$ and $n_2 = 3$. Then both n_1 and n_2 are odd, while their difference $n_1 - n_2 = 4$ is even.

(d) Let a and b be integers. If $a + b$ is even, then either a or b is even.

SOLUTION: False: Counterexample: Let $a = b = 1$. Then $a + b$ is even, but both a and b are odd.

(e) For all integers a, if $a > 2$ then $a^2 - 4$ is composite.

SOLUTION: False: Counterexample: Let $a = 3$. Then $a^2 - 4 = 5$, which is prime.

5. For each of the statements below state whether it is True or False. If True then give a proof. If False then explain why, *e.g.*, by giving a counterexample.

(a) For all positive $x, y \in \mathbb{R}$, if x is irrational and y is irrational then $x + y$ is irrational.

SOLUTION: False: Counterexample: Let $x_1 = \frac{7}{4} - \sqrt{2}$ and $x_2 = \frac{7}{4} + \sqrt{2}$. Then both x_1 and x_2 are positive and irrational, while their sum is $\frac{7}{2}$, which is rational.

For completeness we must also demonstrate that x_1 and x_2 are indeed irrational. This is most easily done by contradiction: Suppose $x_1 = \frac{7}{4} - \sqrt{2}$ is rational. Then $x_1 = \frac{7}{4} - \sqrt{2} = \frac{p}{q}$, for positive integers p and q. It follows that $\sqrt{2} = \frac{7}{4} - \frac{p}{q} = \frac{7q-4p}{4q}$, which is rational. But $\sqrt{2}$ is known to be irrational, and thus we have a contradiction. The proof that x_2 is irrational is very similar to that for x_1. (The fact that $\sqrt{2}$ is irrational can be demonstrated by a proof similar to the one in Problem 5(c) below.)

(b) $\forall x, y \in \mathbb{R}$, if x is irrational and y is rational then $x - y$ is irrational.

SOLUTION: True: We can most easily prove this by contradiction: Suppose x is irrational, y is rational, but $x - y$ is rational. Then $y = \frac{p}{q}$ and $x - y = \frac{r}{s}$, for integers p, q, r, s with $q \neq 0$ and $s \neq 0$. Thus $x = (x-y)+y = \frac{r}{s} + \frac{p}{q} = \frac{rq+ps}{sq}$, which is rational, so that we have a contradiction.

(c) $\sqrt{3}$ is irrational.

SOLUTION: True: We prove this by contradiction: Suppose that $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{p}{q}$, for positive integers p and q. By factoring out common factors we may assume that the only common divisor of p and q is 1. Now from $\sqrt{3} = \frac{p}{q}$ it follows that $p^2 = 3q^2$. Thus $3|p^2$. It is easily seen that therefore $3|p$ (see below). Thus $p = 3k$ for some positive integer k. Hence $p^2 = (3k)^2 = 3q^2$, from which it follows that $3k^2 = q^2$. Thus also $3|q^2$. Again this implies that $3|q$. Therefore we have found that both p and q are divisible by 3, which contradicts the fact that their only common divisor is 1.

For completeness we prove a fact that was used two times in the above proof, namely that $3|p^2$ implies $3|p$. This is most easily done by proving the contrapositive: If p is not divisible by 3 then p^2 is not divisible by 3. So assume p is not divisible by 3. Then p can be written as $p = 3k + 1$ or $p = 3k + 2$. In the first case this gives $p^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, which is not divisible by 3. In the second case we have $p^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, which is not divisible by 3 either.

(d) $\log_5(2)$ is irrational.

SOLUTION: True: This is most easily proved by contradiction: Suppose $\log_5(2)$ is rational. Then $\log_5(2) = \frac{p}{q}$ for positive integers p and q. By definition of the logarithm function this means that $2 = 5^{p/q}$, or equivalently, $2^q = 5^p$. Now 2^q is even, while 5^p

is odd, so they cannot be equal. Hence we have a contradiction. (That 2^q is even is clear, since $2|2^q$. Also is is clear that 5^p is odd, for if it were even then it would be divisible by 2, which is clearly not the case.)

6. (a) Prove that for any $n \in \mathbb{Z}$, $n(n^2-1)(n+2)$ is divisible by 12 .

   PROOF: The expression can be rewritten as $(n-1)n(n+1)(n+2)$, that is, the product of four consecutive integers. Now it is clear that of any four consecutive integers there are two that are even and at least one that is divisible by 3 (You may prove this!) Thus $(n-1)n(n+1)(n+2)$ is divisible by 12.

   (b) Prove $n \in \mathbb{Z}^+$ is divisible by 9 if and only if the sum of its digits is divisible by 9.

   PROOF: Written in decimal form n has the form $[d_n d_{n-1} \cdots d_2 d_1 d_0]$, by which we mean that

$$n = d_0 + 10d_1 + 100d_2 + \cdots + 10^{n-1}d_{n-1} + 10^n d_n \ .$$

   This can be rewritten as $n = S_d + S_9$ where

$$S_d = d_0 + d_1 + d_2 + \cdots + d_{n-1} + d_n \ , \quad S_9 = 9d_1 + 99d_2 + 999d_3 + \cdots + 999\cdots 9d_n \ .$$

   Clearly $9|S_9$. Hence if also $9|S_d$ then $n = S_d + S_9$ is divisible by 9.

   On the other hand, if we assume that $9|n$, and knowing that $9|S_9$, then it follows that $S_d = n - S_9$ is divisible by 9.

7. Give a direct proof that the geometric sum $G(n,a)$, defined as $G(n,a) = \sum_{k=0}^{n} a^k$, where $n \in \mathbb{Z}^+$ and $a \in \mathbb{R}$, with $a \neq 0$ and $a \neq 1$, is equal to $G(n,a) = (1-a^{n+1})/(1-a)$.

   Hint: Write down both $G(n,a)$ and $a \cdot G(n,a)$ in expanded form (without summation sign) and subtract the latter from the former.

   Why are the cases with $a = 0$ and $a = 1$ excluded?

   PROOF:

$$G(n,a) = 1 + a + a^2 + \cdots + a^{n-1} + a^n \ ,$$

   so

$$aG(n,a) = \qquad a + a^2 + \cdots + a^{n-1} + a^n + a^{n+1} \ .$$

   Thus

$$aG(n,a) - G(n,a) = a^{n+1} - 1 \ ,$$

   which can also be written as

$$G(n,a)(a-1) = a^{n+1} - 1 \ ,$$

   from which

$$G(n,a) = \frac{1-a^{n+1}}{1-a} \ .$$

   The case $a = 0$ is excluded as it would give a term 0^0 in the sum $\sum_{k=0}^{n} a^k$ when $k = 0$. The case $a = 1$ is excluded as it would give $\frac{0}{0}$ in $(1-a^{n+1})/(1-a)$. (Both 0^0 and $\frac{0}{0}$ and undefined mathematical expressions.)

8. Let E denote the set of all even positive integers, O the set of all odd positive integers, and, as usual, $\mathbb{Z}^+$ the set of all positive integers. Which of the following expressions are *meaningless*, that is, should *never be used* in your work:

(a) $E \in \mathbb{Z}^+$ (b) $(E \cap \mathbb{Z}^+) \cup O$ (c) $O \Rightarrow \mathbb{Z}^+$ (d) $E \cup O \leftrightarrow \mathbb{Z}^+$

SOLUTION: (a), (c), and (d) are meaningless. Neither the lecture notes nor the text book have given meaning to these espressions which mix logical and set notation illigitimately.

9. For each of the following, determine whether it is valid or invalid. If valid then give a proof. If invalid then give a counter example.

(a) $(A \cap B) \cup (C \cap D) \;=\; (A \cap D) \cup (C \cap B)$

SOLUTION: INVALID: Let $A = \{1\}$, $B = \{2\}$, $C = \{2\}$, and $D = \{1\}$. Then $(A \cap B) \cup (C \cap D)$ is empty, while $(A \cap D) \cup (C \cap B) = \{1, 2\}$, so that these two sets are unequal for this choice of the sets A, B, C and D.

(b) $A - (B \cup C) = (A - B) \cap (A - C)$

SOLUTION: VALID:

$$
\begin{aligned}
x \in A - (B \cup C) \;&\Leftrightarrow\; x \in A \;\wedge\; x \notin (B \cup C) \\
&\Leftrightarrow\; x \in A \;\wedge\; \neg(x \in B \vee x \in C) \\
&\Leftrightarrow\; x \in A \;\wedge\; (x \notin B \wedge x \notin C) \\
&\Leftrightarrow\; x \in A \;\wedge\; x \notin B \;\wedge\; x \in A \;\wedge\; x \notin C \\
&\Leftrightarrow\; x \in (A - B) \;\wedge\; x \in (A - C) \\
&\Leftrightarrow\; x \in (A - B) \cap (A - C).
\end{aligned}
$$

(c) $B \cap C \subseteq A \;\Rightarrow\; (C - A) \cap (B - A)$ is empty

SOLUTION: VALID: We can prove this by contradiction: Suppose $B \cap C \subseteq A$, but $(C - A) \cap (B - A)$ is not empty. Then there exists an element $x \in (C - A) \cap (B - A)$. Thus this x satisfies $x \in C$, $x \in B$ and $x \notin A$. But this contradicts the assumption that $B \cap C \subseteq A$.

(d) $(A \cup B) - (A \cap B) = A \;\Rightarrow\; B$ is empty

SOLUTION: VALID: We prove this by contradiction: Suppose $(A \cup B) - (A \cap B) = A$, but B is not empty. Then there exists an element $x \in B$. There are two possible cases:

i. x is also an element of the set A. Then $x \in A \cup B$ and $x \in A \cap B$. Hence $x \notin (A \cup B) - (A \cap B)$. Since $(A \cup B) - (A \cap B) = A$ it follows that $x \notin A$ which is a contradiction.

ii. x is not an element of the set A. Then $x \in A \cup B$ and $x \notin A \cap B$. Hence $x \in (A \cup B) - (A \cap B)$. Since $(A \cup B) - (A \cap B) = A$ it follows that $x \in A$ which is a contradiction.