

CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP232 MATHEMATICS FOR COMPUTER SCIENCE

ASSIGNMENT 1 SOLUTIONS FALL 2012

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

SOLUTION (omitting details):

- (a) $((p \vee r) \wedge (q \vee r)) \leftrightarrow (p \wedge q) \vee r$ TAUTOLOGY
(b) $(p \wedge (\neg(\neg p \vee q))) \vee (p \wedge q)$ CONTINGENCY
(c) $(p \wedge (\neg q \rightarrow \neg p)) \rightarrow q$ TAUTOLOGY
(d) $((p \rightarrow r) \vee (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$ CONTINGENCY

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give a proof *without* using a truth table.

- (a) $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (\neg p \vee r)$
(b) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv ((p \wedge q) \rightarrow r)$
(c) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv (p \rightarrow (q \wedge r))$
(d) $((p \vee q) \wedge (\neg p \vee r)) \equiv (q \vee r)$

SOLUTION:

(a) VALID: $p \rightarrow (q \rightarrow r) \equiv \neg p \vee (q \rightarrow r) \equiv \neg p \vee (\neg q \vee r) \equiv \neg q \vee (\neg p \vee r) \equiv q \rightarrow (\neg p \vee r)$.

(b) INVALID: If $p = T$, $q = F$, and $r = F$ then the LHS is False, while the RHS is True.

(c) VALID: $(p \rightarrow q) \wedge (p \rightarrow r) \equiv (\neg p \vee q) \wedge (\neg p \vee r) \equiv \neg p \vee (q \wedge r) \equiv p \rightarrow (q \wedge r)$.

(d) INVALID: If $p = T$, $q = T$, and $r = F$ then the LHS is False, while the RHS is True.

3. Write down the negations of each of the following statements *in their simplest form (i.e., do not simply state "It is not the case that...")*. Below, x denotes a real number, $x \in \mathbb{R}$.

- (a) The plane is early or my watch is slow.
(b) Doing the assignments is a sufficient condition for John to pass the course.

- (c) If x is positive, then x is not negative and x is not 0.
- (d) $(0 < x \leq 1) \vee (-1 < x < 0)$

- (a) The plane is on time or late and my watch has the correct time or is fast.
- (b) This is of the form $a \rightarrow p$. The negation is $\neg(a \rightarrow p)$ which is equivalent to $a \wedge \neg p$. Thus, in plain language, the negation is “*John does the assignments but does not pass the course*”.
- (c) This is of the form $p \rightarrow (\neg n \wedge \neg z)$. The negation is $\neg(p \rightarrow (\neg n \wedge \neg z))$ which is equivalent to $\neg(\neg p \vee (\neg n \wedge \neg z))$, which in turn is equivalent to $p \wedge (n \vee z)$. Thus, in plain language, the negation is “ *x is positive, and x is negative or zero*” (which for numbers is impossible!).
- (d) The negation is that x does not lie in the half-open interval $(0, 1]$ and x does not lie in the open interval $(-1, 0)$, which is equivalent to saying that $x = 0$ or $x \leq -1$ or $x > 1$.

4. Write the following statements in predicate form, using logical operators \wedge , \vee , \neg , and quantifiers \forall , \exists . Below \mathbb{Z}^+ denotes all positive integers.

- (a) For any $x, y \in \mathbb{Z}^+$ the equation $x^2 + y^2 - z = 0$ has a solution $z \in \mathbb{Z}^+$.
- (b) The equation $x^3 + y^3 = z^3$ has no solutions $x, y, z \in \mathbb{Z}^+$.
- (c) There is no greatest positive integer.
- (d) The difference between two positive integers can be arbitrarily large.

SOLUTION:

- (a) $\forall x, y \in \mathbb{Z}^+ \exists z \in \mathbb{Z}^+ : x^2 + y^2 - z = 0$.
- (b) $\neg \exists x, y, z \in \mathbb{Z}^+ : x^3 + y^3 = z^3$, or equivalently, $\forall x, y, z \in \mathbb{Z}^+ : x^3 + y^3 \neq z^3$.
- (c) $\forall x \in \mathbb{Z}^+ \exists y \in \mathbb{Z}^+ : y > x$.
- (d) $\forall d \in \mathbb{Z}^+ \exists x, y \in \mathbb{Z}^+ : y - x > d$.

5. Five persons, anonymously known as P_1, P_2, P_3, P_4, P_5 , are suspected of being involved in a crime. Suppose we have the following information:

- (a) If P_5 is involved then so is P_3 .
- (b) If P_2 is involved then so are P_5 and P_1 .
- (c) Either P_1 or P_2 , or both, are involved.
- (d) Either P_3 or P_4 , but not both, are involved.
- (e) P_4 and P_1 are either both involved or neither is.

Determine which persons were involved in the crime. Explain your reasoning.

SOLUTION:

Let p_i denote “person P_i was involved”, for $i = 1, 2, 3, 4, 5$. Then the above statements can be written compactly as logical statements

$$(a) \quad p_5 \rightarrow p_3, \quad (b) \quad p_2 \rightarrow (p_5 \wedge p_1), \quad (c) \quad p_1 \vee p_2, \quad (d) \quad p_3 \oplus p_4, \quad (e) \quad p_4 \leftrightarrow p_1,$$

where all five statements are assumed to be True.

From (d) we have two possibilities, namely, $p_3 = T, p_4 = F$, or $p_3 = F, p_4 = T$. We consider each of these possibilities separately.

- (a) If $p_3 = T$ and $p_4 = F$ then it follows from (e) that $p_1 = F$. Since $p_1 = F$ it follows from (c) that $p_2 = T$. Since $p_2 = T$ it follows in particular from (b) that $p_1 = T$. So p_1 is both True and False, which is a contradiction. Thus the starting assumption that $p_3 = T, p_4 = F$ cannot be true, in other words, the assumption that $p_3 = T, p_4 = F$ does not lead to a solution of the crime problem.
- (b) If $p_3 = F$ and $p_4 = T$ then it follows from (a) that $p_5 = F$. Since $p_5 = F$ it follows from (b) that $p_2 = F$. Since $p_2 = F$ it follows from (c) that $p_1 = T$. Finally, since both $p_4 = T$ and $p_1 = T$, we see that (e) is also satisfied.

Having considered the two cases above, we see that there is only one solution, namely p_1 and p_4 were involved, but p_2, p_3 , and p_5 are not.

6. Let $f(x, y, z) = x - y^2 + z^3$, where $x, y, z \in \mathbb{Z}^+$. For each of the following determine its truth value. Justify your answers.

- (a) $\exists x, y, z \quad f(x, y, z) = 0$
- (b) $\forall y \exists x, z \quad f(x, y, z) < 0$
- (c) $\forall y, z \exists x \quad f(x, y, z) < 0$
- (d) $\forall z \exists x, y \quad f(x, y, z) = 0$

SOLUTION:

- (a) TRUE. Let $y = 2, z = 1$ and $x = 3$.
- (b) FALSE. If $y = 1$ we would need $x - 1 + z^3 < 0$, that is, $x + z^3 < 1$. But this is impossible, since both x and z must be greater than or equal to 1.
- (c) FALSE. If $y = 1$ and $z = 1$ then we would need $x < 0$, which is not allowed since $x \in \mathbb{Z}^+$.
- (d) TRUE. Given any $z \in \mathbb{Z}^+$, choose y big enough so that $y^2 > z^3$, and set $x = y^2 - z^3$. Then x is indeed positive, and we have that $x - y^2 + z^3 = 0$.

7. Let P and Q be predicates on the set S , where S has two elements, say, $S = \{a, b\}$. Then the statement $\forall xP(x)$ can also be written in full detail as $P(a) \wedge P(b)$. Rewrite each of the statements below in a similar fashion, using P , Q , and logical operators, but without using quantifiers.

- (a) $\forall x, y(P(x) \vee Q(y))$
- (b) $\exists xP(x) \wedge \exists xQ(x)$
- (c) $\exists x, y(P(x) \wedge Q(y))$
- (d) $\forall x\exists y(P(x) \wedge Q(y))$

SOLUTION:

$$(a) \quad \forall x, y(P(x) \vee Q(y)) \equiv (P(a) \vee Q(a)) \wedge (P(a) \vee Q(b)) \wedge (P(b) \vee Q(a)) \wedge (P(b) \vee Q(b)).$$

$$(b) \quad \exists xP(x) \wedge \exists xQ(x) \equiv (P(a) \vee P(b)) \wedge (Q(a) \vee Q(b))$$

$$(c) \quad \exists x, y(P(x) \wedge Q(y)) \equiv (P(a) \wedge Q(a)) \vee (P(a) \wedge Q(b)) \vee (P(b) \wedge Q(a)) \vee (P(b) \wedge Q(b)).$$

$$(d) \quad \forall x\exists y(P(x) \wedge Q(y)) \equiv \left[(P(a) \wedge Q(a)) \vee (P(a) \wedge Q(b)) \right] \wedge \left[(P(b) \wedge Q(a)) \vee (P(b) \wedge Q(b)) \right]$$

8. For each of the following equivalences, determine if it is valid for all predicates P and Q . If yes then give a full explanation. If not then provide a counterexample.

- (a) $\exists x(P(x) \vee Q(x)) \equiv \exists xP(x) \vee \exists xQ(x)$
- (b) $\exists x(P(x) \wedge Q(x)) \equiv \exists xP(x) \wedge \exists xQ(x)$
- (c) $\forall x, y(P(x) \wedge Q(y)) \equiv \forall xP(x) \wedge \forall xQ(x)$
- (d) $\forall x, y(P(x) \vee Q(y)) \equiv \forall xP(x) \vee \forall yQ(y)$

SOLUTION:

(a) VALID :

If the LHS is True then there is an x , say, x^* such that (i) $P(x^*) = T$ or (ii) $Q(x^*) = T$. In case (i) $\exists xP(x)$ is True, while in case (ii) $\exists xQ(x)$ is True, so that in both cases the RHS is True,

If the RHS is True, then (i) there is an x , say, x_1 such that $P(x_1) = T$, so that the LHS is also True with $x = x_1$, or (ii) there is an x , say, x_2 such that $Q(x_2) = T$, so that the LHS is also True with $x = x_2$.

(b) INVALID : Take for example the predicates $P(x) = \text{"}x \text{ is even"}$ and $Q(x) = \text{"}x \text{ is odd"}$ on the positive integers. Then the LHS is False, while the RHS is True.

(c) VALID :

If the RHS is True then $P(x)$ is True for all x and $Q(x)$ is True for all x , so that on the LHS $P(x) \wedge Q(y)$ is True for any x and y .

If on the other hand the RHS is False then we must consider two cases, namely (i) there is an x , say x_1 , for which $P(x_1) = F$. In this case the LHS is also seen to be False by taking x to be x_1 and any choice of y . Case (ii) is when there is an x , say x_2 , for which $Q(x_2) = F$. In this case the LHS is also seen to be False by taking y to be x_2 and any choice of x .

(d) VALID :

If the RHS is True then we must consider two cases: (i) P is always True or (ii) Q is always True. In either case the LHS is True also.

If on the other hand the RHS is False then there is an x , say x_1 , such that $P(x_1) = F$ and there is an y , say y_1 , such that $Q(y_1) = F$. Then the LHS is seen to be False also, namely by picking $x = x_1$ and $y = y_1$.